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THE

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PREFACE.

THE Science of Force, or Dynamics, lays the foundation for definite and accurate knowledge in all other Physical Sciences. It supplies the fundamental standards by means of which natural phenomena may be treated as measurable quantities, and the methods by which they may be analysed and compounded; and therefore the student of science has to seek an acquaintance with its principles at an early stage of his course. The object of the present work is to present and unfold the subject according to the method best suited for the purpose of the science student. It aims at supplying, without the aid of advanced mathematics, such explanations of the Laws of Dynamics as will prepare the way for their application to physical phenomena, particularly to those of Heat and Electricity.

For example, the relations of the fundamental units of time, space, and mass, and the method of transferring expressions from one system of absolute units to another, by means of their dimensions, are fully explained.

The *graphic method*, so useful in Physics, of representing quantities which depend on the product of two variables by geometrical areas, is employed throughout.

All the properties of forces, as well as the relation of force and mass, are developed from Newton's Laws of Motion, so that the Principles of Statics appear as particular cases of more general laws.

The law of the Conservation of Energy is applied to investigate the motion of machines and systems of rigid bodies.

Some of the Examples and Exercises have been collected from University Examination Papers, others have been framed to illustrate the chapters to which they are appended. No Exercises, however, will be found to present greater difficulties than the illustrative examples worked out on the same subject. Questions on motion are given in connection with every chapter; for example, the principal question suggesting itself respecting a machine is not what power and weight will keep it *at rest*, but, given a power and weight, what will be the *motion?* Or the converse, given a certain motion, what relation between the power and weight will produce it?

I am greatly indebted to Professor FOSTER, Professor LAMBERT, Mr. H. COURTHOPE BOWEN, and other friends, for suggestions and corrections of proof-sheets.

CONTENTS.

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PRINCIPLES OF DYNAMICS.

CHAPTER I.

TIME AND SPACE.

1. Introduction.—Of the simple ideas which form the basis of our investigations and speculations in Physical Science, the first, or most fundamental, are duration and extension, or time and space. It is sufficient for our purpose, without attempting to define them, to remark that time and space are conceived by us as unbounded, continuous, and divisible without end. Absolute time and absolute space may be conceived as existing without matter, for if the whole universe of matter were destroyed, time and space would remain. Neither, however, would be capable of measurement without the existence of matter. Portions of space can be considered only by reference to material objects, and portions of time only by means of motion.

2. The Measurement of Space and Time.—The basis of measurement is the test of equality afforded by the axiom, "Things which are equal to the same thing are equal to one another."

Spaces are measured by dividing them into parts, each equal to a known or standard space. Similarly any magnitude is measured by finding how many times a certain known or standard magnitude of the same kind is contained

in the given magnitude. The standard magnitude is termed a unit.

Extension is of three kinds, linear, superficial, and solid, and a unit of each kind is required for their measurement.

Nature supplies us with no units which are clearly distinguishable, and by means of which lengths, surfaces, or volumes, can be readily compared with one another. Men are therefore compelled to select arbitrary standards, and it is not surprising that different nations have chosen different standards.

3. Units of Space.—The English standard of length is the *yard*, which is defined by Act of Parliament as the distance between the centres of two gold plugs in a bronze bar deposited in the Exchequer, the bar being at the temperature 62° F.

The third part of the standard yard, termed the *foot*, is the usual English scientific unit of length. The twelfth part of the foot is the *inch*.

The French unit of length is the *metre*, defined in French law as the length of a certain rod of platinum at the temperature of melting ice. The metre is nearly equal to 39·37079 inches.

Other nations have different legal standards of length, but the importance to scientific men that there should be a common understanding of the measures to be employed has led to the adoption of the metre as the standard in many countries of Europe. It will not be necessary for us to use other units of length than the foot, the metre, and multiples and parts of the foot and metre.

Units of surface and volume may be derived from the unit of length. It is true they might be chosen independently, but it is of very great scientific and practical importance that only the necessarily independent fundamental standards shall be arbitrarily selected, and that all

others shall be deduced from these standards in a systematic manner. This principle applies not only to quantities so nearly allied as length, surface, and volume, but to every kind of quantity whatever, and it must be constantly borne in mind in the investigation on which we are entering.

4. Units of Time.—There is more uniformity throughout the world in the selection of units of time, for Nature has furnished all men alike with three natural units of time—the period of the earth's rotation round its axis, the period of the moon's revolution round the earth, and the period of the earth's revolution round the sun. The first of these, or the *day*, is the only one we have here to consider. The interval between successive transits of the same star is termed a *Sidereal day*. It is the true period of the earth's rotation. It can be ascertained with great exactness by the ordinary observations of astronomers, and is, as far as we know, invariable. The interval between successive transits of the sun is termed the *Solar day*. A solar day is a little longer than a sidereal day, and varies by a very small range during the year. The average length of a solar day can be accurately deduced from the sidereal day and the year. If a perfectly regular clock were to keep time according to the average length of the solar day, so as to be a little in advance of solar time after the solar days have been shortest, and a little in arrear after the solar days have been longest, coinciding with solar time at certain epochs in the year, the day, as marked by the clock, would be what is called the mean solar day. In all countries of the world the solar day is the principal unit of time, and its mean value is, as far as we know, invariable. The 86,400th part of the mean solar day, or the *second*, is the unit of time ordinarily used in scientific investigations.

5. Position.—Situation in space is termed position. We have no means of determining position absolutely; and any

position can be described only by referring it to better known positions by means of the units of space. The position of a point is determined by its direction and distance from any known point; in other words, the elements of its position are direction and distance.

Hence the position of a point may be expressed—

(i.) If it be on a known or fixed line, by its distance from a fixed point on that line.

(ii.) If it be in a known plane, by its co-ordinates or distances from two fixed intersecting lines in that plane (usually for convenience taken at right angles).

(iii.) If it be in free space, by its distances from three fixed intersecting planes (usually at right angles).

The position of a straight line is given by—

(i.) The positions of two points in it, or

(ii.) The position of one point in it and its direction.

The position of a plane is usually given by—

(i.) The positions of three points in the plane.

(ii.) The positions of a line and a point not in the line.

(iii.) The positions of two lines in it.

(iv.) The position of a point in it and the direction of a line perpendicular to it.

CHAPTER II.

MOTION.

SECTION I.—*Motion in a Straight Line.*

6. Introduction.—Besides time and space, we are cognisant of three elementary ideas, to one or other of which every distinct conception connected with the world of Nature must belong—namely, motion, matter, and force; and the first step in preparing the way for an exact knowledge of physical science is to obtain a clear understanding of the nature of these things, and their relations to one another.

The present chapter is devoted to the consideration of the simplest kinds of motion independently of force and matter.

7. Definition of Motion.—*Motion* is change of position. It connects time and space. A point is at *rest* when it occupies always the same position, and is in *motion* when it occupies different positions at different times.

In speaking of a moving point we may, for definiteness, suppose it to be the position of a small body termed a particle, whose dimensions are so small that they need never be taken into account.

8. Velocity.—Rate of motion is termed *velocity.* It may be uniform or variable.

Velocity is *uniform* when equal spaces are passed over in equal portions of time; and it is *variable* when the point moves over unequal spaces in equal portions of time.

When uniform, the velocity is measured by the space passed over in a unit of time. Usually for scientific purposes velocity is measured in feet per second.

When the velocity is variable, any definite value of it must be associated with a particular instant in time, or a particular position of the particle in space. In this case the velocity is the space which would be passed over in a unit of time by the particle if this velocity were maintained throughout the unit of time.

The velocity of a particle moving uniformly may be found by dividing the length of any portion of its path by the time taken to move through that portion. When the motion is variable, we may approximate to the velocity of a particle at any instant by taking an interval of time indefinitely short and dividing the space described in this time, by the time. Thus, if s be the space passed over in an interval of time t so small that we may consider the velocity to be sensibly unaltered during it, then the velocity at the instant from which t is measured is $\frac{s}{t}$.

An example may make this statement plainer. Suppose a locomotive to travel between two given places, so that if a certain length be taken (as, for instance, 100 yards) in any part of the distance, it will be passed over in the same time, as, for instance, in $14\frac{2}{3}$ seconds, then, in this case the locomotive has a uniform velocity which may be expressed as 100 yards in $14\frac{2}{3}$ seconds, or 880 yards in a minute, or 30 miles an hour, or 44 feet per second. Now, suppose that between two other places the speed continually changes, but that at a certain point the locomotive passes over 50 yards in

6 seconds, and the change of velocity in this time is inappreciable; then the velocity at the given place is 50 yards ÷ 6, or 25 feet per second, or, which is the same thing, $17\frac{1}{22}$ miles per hour. If the time 6 seconds be not so small that the change of velocity during it is inappreciable, then a smaller portion of time must be taken, and the distance traversed in this time, divided by the time, must be taken as the measure of the velocity.

9. Acceleration.—The rate of variation of velocity is termed *acceleration*. It is uniform or variable.

Acceleration is uniform when there are equal increments of velocity in equal intervals of time, and uniform acceleration is measured by the increase of velocity in a unit of time.

When variable, the acceleration at any instant of time or at any position of the particle in space is measured by the amount by which the velocity would be increased in a unit of time if the same rate of increase were maintained throughout this time.

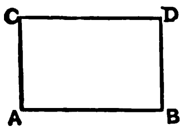
When uniform, the acceleration is the gain of velocity in any time divided by the time; and, as in the case of velocity, variable acceleration may be measured at any point by supposing it constant through a very small interval of time. Hence if v be the increase of velocity in a very small interval of time, t, the acceleration is $\frac{v}{t}$.

10. Graphic Representation of Space Passed Over.—When a particle is moving with uniform velocity, the space passed through in any second may be represented by the constant velocity v, and the space in any number of seconds, t, will then be vt.

Now, as this space is the product of two quantities, both of which may be represented by straight lines, it follows that

it can be represented by the rectangle contained by these lines.

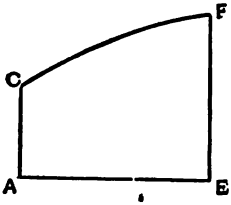
For example, let the lines AB and AC represent t and v respectively, that is to say, let the number of units



of length in AB be equal to the number of units of time in t, and the number of units of length in AC be equal to the number v; then the area of the rectangle ABDC will contain vt units of area. But we have seen that the number which represents the space passed over is vt; hence the space passed over and the area are represented by the same number.

This fact will enable us readily to represent the space passed over in a given time t, by a body moving with variable velocity when the law of variation or the actual velocity at every instant is known.

11. **Proposition I.**—If a particle P moves so that the



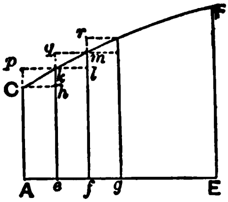
straight line AE represents the time of motion, AC the velocity at the commencement of the time, and the ordinate of the curve CF at any point in AE represents the velocity at the corresponding instant of time; then the space passed over will be represented by the area AEFC.

Let the time be divided into small portions represented by Ae, ef, fg, etc.

Let us compare the motion of the particle P with that of two imaginary particles M and N, which move so that their velocities remain constant throughout the intervals of time represented by Ae, ef, fg, etc.

Let the first particle M have a velocity equal to that of our particle P at the *beginning*, and the second particle N

the same velocity as P at the *end*, of each interval. The velocity of M will then be represented by AC, from A to *e*; by *ek*, from *e* to *f*; by *fm*, from *f* to *g*, and so on. The velocity of N will be represented by A*p* or *ek*, from *a* to *e*; by *eq* or *fm*, from *e* to *f*, and so on.



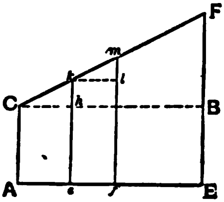
Now it is evident that the space passed over by the particle P is greater than that of M and less than that of N, for, except at the instants when the velocities coincide, the velocity of P is always greater than that of M and less than that of N.

But the space passed over by M is represented by the sum of the rectangles C*e*, *kf*, *mg*, etc., and the space passed over by N is represented by the sum of the rectangles *pe*, *qf*, *rg*, etc.; hence, however small the divisions A*e*, *ef*, may be, the area which represents the space passed over by P lies between the sums of the two sets of rectangles.

Now let the number of intervals in the time AE be increased indefinitely, then the two sums approach indefinitely near to one another, and therefore also to the area AEFC, which always lies between them. Hence the area AEFC must represent the space passed over by the particle P.

12. Uniform Acceleration.—The velocity is said to be uniformly accelerated when it is increased in equal intervals of time by a constant quantity.

Let A*e*, *ef*, etc., represent equal intervals of time, and let AC, *ek*, *fm*, etc., represent the velocities at the instants A, *e*, *f*, etc. Hence *hk*, *lm*, etc., represent the accelerations in these intervals. When the acceleration is uniform, $hk = lm =$ etc., and therefore CF is a straight line.



Let the acceleration be denoted by the letter *f*.

Now let us take Ae, ef, etc., each to represent a second, then hk, lm, etc., are each equal to the acceleration f. Hence if t be the number of seconds in AE, then $BF = ft$.

Let u be the velocity at the instant A and v that at E, then

$$EF = AC + BF$$

$$\text{or } v = u + ft \quad . \quad . \quad . \quad . \quad (1)$$

The space, s, passed over in the time t is represented by the area AEFC. This may be expressed in several ways.

(i.) $$s = \text{rectangle AEBC} + \text{triangle CBF}$$
$$= AC \times AE + \tfrac{1}{2} CB \times BF$$
$$= ut + \tfrac{1}{2} ft^2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(ii.) If G be the middle point of AE, and GH represent the velocity at the instant G, then GH is the arithmetical mean between AC and EF; that is, $GH = \frac{1}{2}(u+v)$.

But the area $AEFC = AE \times GH$;

$$\therefore s = \tfrac{1}{2}(v+u)\,t \quad . \quad . \quad . \quad . \quad (3)$$

(iii.) Since from (1)

$$t = \frac{v-u}{f},$$

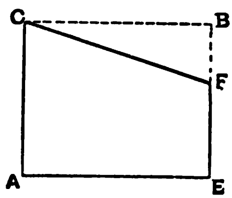
by substituting in the last equation we obtain

$$s = \frac{v^2 - u^2}{2f}$$

$$\text{or } v^2 = u^2 + 2fs \quad . \quad . \quad . \quad . \quad (4)$$

13. Decreasing Velocity.—If the velocity diminish uniformly, the formulæ of motion may be obtained from the above by making f negative.

For in this case we have



$$EF = AC - BF$$
$$\text{or } v = u - ft.$$
$$s = \text{rect. AEBC} - \triangle CFB,$$
$$= ut - \tfrac{1}{2} ft^2$$
$$\text{and } s = \tfrac{1}{2}(u+v)t$$
$$= \frac{u^2 - v^2}{2f}$$
$$\text{or } v^2 = u^2 - 2fs.$$

14. Collected Formulæ.—Hence we have the following equations connecting the velocity, time of motion, and space passed over by a particle moving with constant acceleration.

$$\left.\begin{array}{ll}(1) & v=u\pm ft\\(2) & s=ut\pm\tfrac{1}{2}ft^2\\(3) & s=\tfrac{1}{2}(u+v)t\\(4) & v^2=u^2\pm 2fs\end{array}\right\}A$$

Equation (3) shows that the space passed over in the time t when the acceleration is uniform is equal to that which would be passed over in the same time by a particle moving with constant velocity equal to the mean.

If the time be reckoned from the instant at which the particle is at rest these equations evidently become

$$\left.\begin{array}{ll}(1) & v=ft\\(2) & s=\tfrac{1}{2}ft^2\\(3) & s=\tfrac{1}{2}vt\\(4) & v^2=2fs\end{array}\right\}B$$

From equation (2) we see that if $t=1$ second, then $f=2s$, that is to say, *the acceleration is equal to twice the space passed over in the first second if the particle starts from rest.*

We may find the space passed over in any, the n^{th} second, by subtracting that passed over in $(n-1)$ seconds from that passed over in n seconds;

hence space in the n^{th} second $=\tfrac{1}{2}fn^2-\tfrac{1}{2}f(n-1)^2$

$$=\tfrac{1}{2}f(2n-1)$$

$$=\tfrac{1}{2}f\times \text{ the } n^{th} \text{ odd number.}$$

This reasoning applies whatever the unit of time, hence the space passed over in successive equal intervals reckoned from the instant of rest will be found by multiplying the space passed through in the first of these intervals by the numbers 1, 3, 5, 7, etc., in succession.

15. Units of Velocity and Acceleration.—If we vary our units of length and time, we change the units of velocity and acceleration.

The space passed over in time t when the velocity is v is given by the equation

$$s=vt.$$

Hence v must be 1 when s and t are each 1; in other words, the unit of velocity is that which will carry a particle over a unit of length in a unit of time.

Again, if v be the velocity generated in time t when the acceleration is f,

$$v=ft.$$

Hence f must be 1 when v and t are each 1; in other words, the unit of acceleration is that which will produce a unit of velocity in a unit of time.

Suppose that units of length and time, which, for convenience of reference we will term respectively a and b, have been chosen, then the corresponding units of velocity and acceleration c and d are determined. Now, if l times a and t times b be taken as the units, the new unit of velocity will be v times the old unit, where $v=l \div t$ and the new unit of acceleration will be f times the old, where $f=v \div t$, or if we eliminate v,

$$f=\frac{l}{t^2}.$$

The unit of velocity, therefore, varies directly as the unit of length and inversely as the unit of time, and the unit of acceleration varies directly as the unit of length, and inversely as the square of the unit of time, and hence the unit of acceleration is affected *twice* by a change in the unit of time.

This is sometimes expressed by saying that the *dimensions*

of the unit of velocity are those of $l \div t$, and the dimensions of acceleration those of $l \div t^2$.

Example 1.—When the units of length and time are respectively the inch and the second, an acceleration is represented by 10. What will be the measure of the same acceleration when the units are respectively the yard and the minute?

First Solution.—Here a velocity of 10 inches per second is gained in one second.

$\therefore$ a velocity of 10×60 inches per second is gained in one minute.

$\therefore$,, 10×60^2 ,, minute ,, ,,

$\therefore$,, $\frac{10 \times 60^2}{3 \times 12}$ yards per ,, ,, ,,

Therefore 1000 is the new measure of the acceleration.

Another Solution.—Since the yard is 36 times the first unit of length, and the minute is 60 times the first unit of time, the ratio of the new unit of acceleration to the old will be

$$f = \frac{l}{t^2} = \frac{36}{3600} = \frac{1}{100}.$$

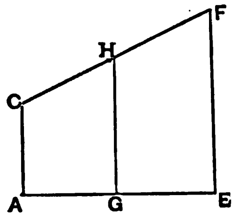
But the numerical measures of a given acceleration in two systems of units are inversely proportional to the units.

Hence the new measure of the given acceleration is 100 times 10, or 1000.

Remark.—It may not be unimportant to direct attention particularly to the above statement, that the numerical measures of a given quantity in two systems of units are inversely as the units. For instance, a sum of 20 guineas may be paid by 21 sovereigns or by 420 shillings; that is, the measure of the sum is 21 when the unit is a sovereign, but 20 times 21 when the unit is the twentieth part of a sovereign. The numerical measure of a day is 24 when the unit is the hour, but when the minute, which is the sixtieth part of the hour, is the unit, the measure of the day is 60 times 24, or 1440. If a given length be 80 times one unit, but 100 times another, since these numbers are as 4 to 5, the units are as 5 to 4.

Example 2.—A particle moves with uniform acceleration for 10 seconds, during which time its velocity increases from 20 feet per

second to 50 feet per second; find the space passed over in the 10 seconds.



Let AE represent 10 secs.
AC „ 20 ft.
EF „ 50 ft.

Therefore if G be the middle point of AE, then GH represents 35 feet, and the area AEFC is AE × GH or 10 × 35.

Hence the space passed over, which is represented by the area AEFC = 350 feet.

Example 3.—A particle moves over 600 feet while its velocity decreases from 80 feet to 10 feet per second; find the time of motion.

Let AE represent the time, let AC represent 80 feet and EF 10 feet, then GH represents 45 feet. Hence 45 times AE is the area of the figure, and therefore

$$45t = 600,$$

$$\therefore \quad t = 13\tfrac{1}{3} \text{ secs.}$$

Example 4.—If the unit of velocity be 8 feet in two seconds, and the unit of acceleration 16 feet per second in a second, find the unit of time.

8 feet in 2 seconds is 4 times 1 foot per second.

16 feet per second per second is 16 times 1 foot per second per second.

Hence if t times a second be the unit of time, the formula $v = ft$ gives

$$4 = 16t$$

$$\therefore t = \tfrac{1}{4}.$$

EXERCISE I.

1. A particle moves with uniform acceleration for 12 seconds, during which its velocity increases from 56 feet to 92 feet per second; find the space passed over.

2. A particle moves with uniform acceleration over $132\tfrac{1}{2}$ feet in five seconds, and has at the end of this time a velocity of 5 feet per second; find the initial velocity and the acceleration.

3. A particle starting with a velocity of 90 feet per second loses of its velocity 5 feet per second in every second of its motion; how far will it move?

4. Two bodies, A and B, move in a certain time over the same space; the velocity of A is uniformly accelerated and the velocity of B is constant. Show that the mean velocity of A is equal to that of B.

5. When the units of time and space are respectively the second and the foot, the acceleration of a moving body is 20; what would be the acceleration if the units were respectively the second and the metre (a metre = 39·37 inches)?

6. In the above case, what would be the measure of the acceleration if the velocity were measured in yards per minute?

7. A train moves with regularly increasing speed for three minutes, and in this time passes over a mile; what is the speed at the end of the three minutes?

8. A body moving with uniform acceleration describes 570 feet in the eighth second; find the acceleration.

9. If a train moves uniformly for ten minutes with a speed of 30 miles an hour, and then for ten minutes with a speed which decreases uniformly in the ten minutes to 15 miles an hour; find the space passed over in the twenty minutes.

10. A body describes 408 feet while its velocity increases from 29 to 73 feet per second; find the whole time of motion and the acceleration.

11. Find the numerical value of the acceleration when in a quarter of a second a velocity is produced which would carry a body over 20 feet in a minute, the units of length and time being the yard and the second.

12. The velocity of a body is diminished regularly by 12 feet per second, and the body comes to rest in 20 seconds; find the velocity of starting.

13. A particle is observed to move over 70 feet and 86 feet in two successive seconds; how far will it move in ten seconds reckoned from the instant at which the velocity is 94 feet per second?

14. Two points, A and B, are 100 yards apart; a particle starts from A towards B with a uniform velocity of 12 feet per second at the same instant as another particle starts from B towards A. The latter particle starts from rest, and its velocity increases at the rate for every second of 12 feet per second. When and where will the particles meet?

15. The measure of an acceleration is 30 when the units of time and space are the minute and mile; find the measure of the same acceleration when the units are respectively the foot and the second.

16. If the unit of velocity be 12 feet in three seconds, and the unit of acceleration 900 miles per hour per hour; find the unit of time.

17. A body passes over 300 feet in 5 seconds, and during the 5 seconds the velocity is doubled; find the acceleration.

18. Having given the measure, v, of a certain velocity with given units l and t of length and time; find the measure v' of the same velocity, with other given units l' and t'.

19. Having given the measure f of a certain acceleration, with given units l and t of length and time; find the measure of the same acceleration, with other given units l' and t'.

20. A train acquires the velocity of 1000 yards per minute in an hour. If the motion be uniformly accelerated, find the measure of the acceleration, taking a foot and a second as the units of space and time.

21. How many times does the acceleration of 120 yards per minute per minute contain the acceleration of one inch per second per second?

22. If a mile per minute be the unit of velocity, and a yard the unit of space; find the unit of time.

23. If 11 yards per second per second be the unit of acceleration, and a minute the unit of time, find the unit of space.

24. Two bodies A and B move with different accelerations from rest; A moves over $(a+b)$ feet in $(s-t)$ seconds, B moves over $(a-b)$ feet in $(s+t)$ seconds. If the units used are for A the inch and second, and for B the yard and minute, the measures of the accelerations are as 2 to 1, and the whole velocities gained by A and B in the above-mentioned times respectively are as 10 to 1; prove that s is to t as 13 to 11, and a is to b as 43 to 7.

SECTION II.—*Composition and Resolution of Velocities and Accelerations.*

16. Velocities in the same Straight Line.—The actual velocity of a body may be composed of two or more velo-

cities along the same straight line, and in the same or opposite directions. For example, a man may walk along the deck of a ship while the ship itself is in motion. Suppose the man be walking and the vessel moving in the same direction and with uniform speed, then it is evident that—

(i.) If he walks in the direction of the motion of the ship, the actual distance through which he moves in a second is the *sum* of the distances moved through by the vessel in the water and the man along the deck.

(ii.) If the directions are opposite, the actual distance is the difference of the distances traversed by the ship with regard to the land and the man with regard to the ship.

(iii.) If the vessel moves forward and the man walks from the bow to the stern with the same speed, he occupies the same position as he would if he stood upon the deck and the vessel did not move.

17. Definition of Resultant and Component.—If a body has two or more velocities simultaneously imparted to it, they are termed *component* velocities, and the actual velocity is termed the *resultant.*

The results in the above illustration may therefore be stated as follows :—

(i.) If the components act in the same direction, their resultant is their sum.

(ii.) If two opposite velocities are simultaneously given to a body, the resultant velocity is their difference and is in the direction of the greater.

(iii.) Two equal and opposite velocities neutralise one another, and the body to which they are imparted remains at rest.

18. Composition of Velocities in Different Directions.—A velocity in one direction may be compounded of two velocities in different directions.

Suppose, for example, that the ship in the preceding

illustration is drifting sideways at the same time that the man in it is walking from one end of the deck to the other. The actual velocity of the man with regard to the land will be produced by two simultaneous velocities in different directions. It will be completely known if we know the direction and rate of motion of the ship and the direction and rate at which the man walks along the deck. If both rates be uniform, the real path of the man will be a straight line. The magnitude and direction of the resultant velocity may be found according to the following proposition, termed the parallelogram of velocities. By its means, treating as an axiom the purport of the above illustration, namely, that the actual velocity of a particle may always be supposed to result from two or more simultaneous velocities in given directions, we are able to compound simultaneous velocities, first, in the same plane; secondly, in different planes; and also to separate any given velocity into two or more components which shall be together equivalent to it. For instance, a body moving along an inclined plane may be supposed to have a velocity u towards the north, v towards the west, and w vertically upwards. This is equivalent to supposing that the body moves in a horizontal plane along a line directed due north with velocity u, while that line moves parallel to itself in a horizontal plane with velocity v, and the plane rises parallel to itself with velocity w.

As acceleration is merely *change of velocity* in a given direction, the rule for the composition of velocities suggests a similar rule for the composition of accelerations; that is to say, a rule for finding the single acceleration which is equivalent to two or more simultaneous accelerations.

Proposition II.—*If two velocities possessed simultaneously by a particle be represented in magnitude and direction by two adjacent sides of a parallelogram, the resultant velocity will be*

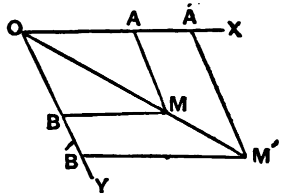
represented in magnitude and direction by the diagonal of this parallelogram drawn through the intersection of these sides. Let a particle at O have simultaneously a velocity v along OX and a velocity v' along OY. Let M be a position of the particle at time t, and suppose MA drawn parallel to OY meeting OX in A. Then the hypothesis with regard to the velocities is equivalent to the supposition that the point M moves along the line AM with velocity v' while this line is displaced parallel to itself, so that its extremity A moves along OX with velocity v.

$$\text{Therefore } \mathrm{OA}=vt \text{ and } \mathrm{AM}=v't$$

$$\text{Hence } \frac{\mathrm{OA}}{\mathrm{AM}}=\frac{v}{v'}.$$

Let M′ be the position of the particle at time t'.
Then also $\mathrm{OA}'=vt'$ and $\mathrm{A'M'}=v't'$.

$$\text{Hence } \frac{\mathrm{OA}}{\mathrm{AM}}=\frac{\mathrm{OA}'}{\mathrm{AM}'}.$$



Suppose now the points M and M′ to be each joined with O, then the above proportion shows that in the triangles OAM, OA′M′ the sides about the equal angles A and A′ are proportional, and therefore the triangles are similar and the angles AOM, A′OM′ are equal. Hence OMM′ is a straight line; that is to say, any two positions of the point M lie on a straight line passing through O.

The same triangles give the proportion

$$\frac{\mathrm{OM}}{\mathrm{OM}'}=\frac{\mathrm{OA}}{\mathrm{OA}'}=\frac{vt}{vt'}=\frac{t}{t'},$$

showing that the space described by the point M is proportional to the time; hence M moves in a straight line, with uniform velocity.

To determine this velocity let t in the above equation be

the unit of time, then OA represents v, AM represents v', and OM the resultant velocity of M. Hence the resultant velocity is the diagonal of the parallelogram constructed on OA, OB, which represent the component velocities.

20. Deductions from the Parallelogram of Velocities.—From this proposition the following deductions may be made:—

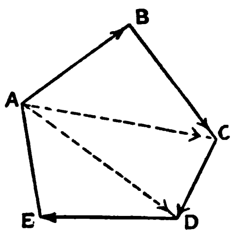
(i.) If the component velocities are at right angles, the resultant is the square root of the sum of their squares.

(ii.) If two velocities are represented by OA and AM in the figure, the resultant is a velocity represented by the third side OM of the triangle OAM. Hence it is not necessary to construct a parallelogram in order to compound two given velocities, a triangle which is half the parallelogram being sufficient.

(iii.) If a particle have given to it simultaneously the velocities represented by the sides OA, AM, MO of a triangle *taken in order*, it remains at rest.

Proof.—For the velocities OA, AM are equivalent to a velocity OM. Add the third velocity MO, then the resultant of the equal and opposite velocities OM, MO is zero.

(iv.) The resultant of velocities represented by the sides of a closed polygon ABCDE taken in order is zero.



Proof.—For AB, BC are equivalent to AC; AC, CD to AD; and the resultant of AD, DE, EA is zero by (iii.).

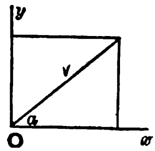
(v.) The resultant of velocities represented by all the sides but one of a closed polygon taken in order is represented by the remaining side, taken in the direction opposed to that of the components.

Proof.—For if a velocity be added equal to the remain-

ing side taken in the same order as the other components, there will be no resultant; hence, if this velocity be reversed, it must be equivalent to all the others.

(vi.) Any velocity may be considered as resulting from two simultaneous velocities, the components being represented by the sides of a parallelogram having the line representing the resultant velocity for diagonal.

It is frequently convenient to suppose a given velocity V to result from two components, v_x and v_y, at right angles to one another, and making respectively angles a and $90°-a$ with V; then the above proposition shows that



$$V^2=v_x^2+v_y^2$$
$$v_x=V.\cos a$$
$$v_y=V.\sin a.$$

These quantities v_x, v_y may be called the components of the velocity V along the axes Ox, Oy.

Hence the component of any velocity V, in a direction making an angle a with that of V, is V $\cos a$ and—

(vii.) When a point moves in a straight line with uniform velocity, the projection of the point on any straight line whatever moves with uniform velocity.

(viii.) If any number of simultaneous velocities v_1, v_2, v_3, etc., . . . v_n, given to the same particle make angles a_1, a_2, a_3, etc., . . . a_n with a fixed straight line Ox, their resultant V may be obtained by resolving each along and perpendicular to the fixed straight line, finding the sum of the components and making the results equal respectively to V_x, V_y, the components of the resultant.

Proof.—For v_1 may be replaced by $v_1 \cos a_1$ and $v_1 \sin a_1$; v_2 may be replaced by $v_2 \cos a_2$ and $v_2 \sin a_2$, and so on.

Substituting these components in each case for v_1, v_2, etc., we have for the velocity along the fixed straight line

$v_1 \cos\alpha_1 + v_2 \cos\alpha_2 + \ldots + v_n \cos\alpha_n = V_x$ and perpendicular to it

$$v_1 \sin\alpha_1 + v_2 \sin\alpha_2 + \ldots + v_n \sin\alpha_n = V_y$$

$$\text{Also } V^2 = V_x^2 + V_y^2.$$

This gives the magnitude of the resultant velocity V.

Now if θ be the angle made by the direction of the resultant velocity with the fixed straight line, then to obtain the direction of the resultant velocity we have

$$\frac{V_y}{V_x} = \frac{V \sin\theta}{V \cos\theta} = \tan\theta.$$

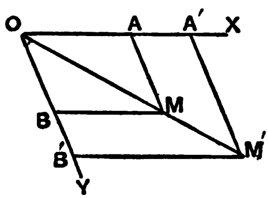
Let the sign Σ before a quantity signify the sum of all the quantities which are like it in form; then the above equations may be written

$$V^2 = (\Sigma v \cos\alpha)^2 + (\Sigma v \sin\alpha)^2.$$

$$\tan\theta = \frac{\Sigma v \sin\alpha}{\Sigma v \cos\alpha}.$$

21. Proposition III.—*If two accelerations imparted simultaneously to a particle be represented in magnitude and direction by two straight lines forming the sides of a parallelogram, then the resultant acceleration will be represented in magnitude and direction by the diagonal of the parallelogram drawn through the point of intersection of these sides.*

Let a particle have simultaneously accelerations f_1 along OA, and f_2 along OB. This is equivalent to supposing the particle to move along OB with acceleration f_2, while OB or its parallel AM moves parallel to itself, the extremity A moving along OA with acceleration f_1. Let OA represent the velocity which would be acquired by A in a time t in consequence of the acceleration f_1, and let OB or AM represent the velocity which would be



acquired in time t along AM in consequence of acceleration f_2. Then the diagonal OM represents the velocity produced in the same time in consequence of both accelerations (Proposition II).

$$\text{Now OA} : \text{AM} :: f_1t : f_2t :: f_1 : f_2.$$

Similarly if we take a longer time, so as to obtain the parallelogram OA'M'B', we have

$$\text{OA}' : \text{A'M}' :: f_1 : f_2$$

$$\therefore \text{OA}' : \text{OM}' :: \text{OA} : \text{OM}.$$

Since the angles at A and A' are equal, and the sides about them proportional, therefore the angles AOM and A'OM' are also equal, and M' is on the straight line OM, whatever may be the time t (Euclid vi. 6). Let the time be a second, so that OA$=f_1$, OB$=f_2$, then OM, which has been shown to have the *direction* of the resultant acceleration, will also represent its *magnitude*.

22. Deductions from the Parallelogram of Accelerations.—Deductions corresponding to those drawn from the parallelogram of velocities may be made here, and may be proved in exactly the same way.

(i.) If the component accelerations are at right angles, the resultant is the square root of the sum of their squares.

(ii.) If two accelerations are represented by OA and AM in the figure, the resultant acceleration is represented by the third side OM of the triangle OMA.

(iii.) Three accelerations *given to the same particle*, and represented by the sides AB, BC, CA, of a triangle ABC, *taken in order*, have a resultant zero.

(iv.) The resultant of accelerations given to the same particle and represented by the sides of a closed polygon taken in order is zero.

(v.) The resultant of accelerations given to the same particle, and represented by all the sides but one of a closed

polygon, taken in order, is represented by the remaining side taken in a direction opposite to that of the components.

(vi.) Any acceleration may be supposed to result from two others represented by the adjacent sides of a parallelogram having the line representing the given acceleration for diagonal. If f be an acceleration, and f_x, f_y its components in two directions at right angles to one another, and making angles a and $90°-a$ respectively, with the direction of f, then

$$f^2=f_x^2+f_y^2$$
$$f_x=f.\cos a$$
$$f_y=f.\sin a.$$

(vii.) When a point moves in any direction with acceleration f, the projection of the point on any line making an angle a with that direction moves with an acceleration $f.\cos a$.

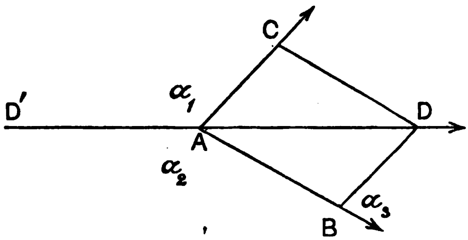
(viii.) If any number of accelerations $f_1, f_2 \ldots f_n$ be given to a particle in directions making angles $a_1, a_2 \ldots a_n$, with a fixed straight line, their resultant f may be obtained by resolving each parallel and perpendicular to the fixed straight line, adding the components in each of the two directions, and equating the results to f_x, f_y, the components of the resultant, then

$$f_x=f_1 \cos a_1+f_2 .\cos a_2+ \ \ldots \ +f_n \cos a_n=\Sigma(f_1 \cos a_1)$$
$$f_y=f_1 \sin a_1+f_2 \sin a_2+ \ \ldots \ +f_n \sin a_n=\Sigma(f_1 \sin a_1)$$
$$f^2=f_x^2+f_y^2=(\Sigma f_1 \cos a_1)^2+(\Sigma f_1 \sin a_1)^2$$
$$\frac{f_y}{f^x}=\frac{f\sin\theta}{f\cos\theta}=\tan\theta=\frac{\Sigma(f_1 \sin a)}{\Sigma(f_1 \cos a)}.$$

(ix.) If two component velocities or two component accelerations be given in magnitude and direction, the resultant can be determined by the solution of a triangle.

Let AB, AC represent component velocities v_1, v_2, and let the angle CAB be a_3. Complete the parallelogram ABDC

and let the resultant velocity AD be v, then a velocity AD′ equal and opposite to AD will exactly neutralise v_1 and v_2 taken together.



Let $\angle CAD' = \alpha_1$ and $\angle BAD' = \alpha_2$.

Consider the triangle ADB: it has its sides respectively proportionate to v, v_1, v_2. Now by a well-known trigonometrical formula (or by Euclid ii. 12, 13),

$$AD^2 = AB^2 + BD^2 - 2AB.BD.\cos ABD$$
$$= AB^2 + BD^2 + 2AB.BD.\cos\alpha_3.$$
$$\therefore\ v^2 = v_1^2 + v_2^2 + 2v_1v_2\cos\alpha_3.$$

Also in any triangle ADB the sides are proportional to the sines of the opposite angles, that is,

$$AD : AB : BD :: \sin ABD : \sin ADB : \sin BAD$$

$$\text{But } \sin ABD = \sin(\pi - \alpha) = \sin\alpha_3$$
$$\sin ADB = \sin(\pi - \alpha_1) = \sin\alpha_1$$
$$\sin BAD = \sin(\pi - \alpha_2) = \sin\alpha_2.$$

$$\text{Hence } \frac{v}{\sin\alpha_3} = \frac{v_1}{\sin\alpha_1} = \frac{v_2}{\sin\alpha_2}.$$

Similarly if f_1, f_2 be two accelerations given to the same particle in directions making an angle α_3 with each other, and if f will exactly neutralise f_1, f_2, when making angles α_2 and α_1 with them respectively, then

$$f^2 = f_1^2 + f_2^2 + 2f_1f_2\cos\alpha_3,$$

and similar equations for f_1^2 and f_2^2.

$$\text{Also } \frac{f}{\sin\alpha_3} = \frac{f_1}{\sin\alpha_1} = \frac{f_2}{\sin\alpha_2}.$$

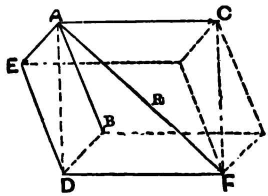
23. Definition of Equilibrium.—Velocities or accelerations which produce no change of motion when simultaneously applied to the same particle are said to be in equilibrium.

The above formula, which is easily remembered on account of its symmetry, shows that when three velocities or three accelerations are in equilibrium, if each be divided by the sine of the angle between the other two, the quantities thus formed will be equal.

24. Proposition IV.—*If three velocities or three accelerations given simultaneously to a particle be represented in magnitude and direction by three adjacent edges of a parallelopiped, their resultant will be represented by the diagonal drawn from the intersection of these edges.*

Let AC, AE, AB, the three edges of a parallelopiped, represent velocities v_1, v_2, v_3, simultaneously given to the same particle; then AF, the diagonal from it, will represent their resultant.

Proof.—Draw AD, CF; then AD represents the resultant of the velocities AE and AB by Prop. III.



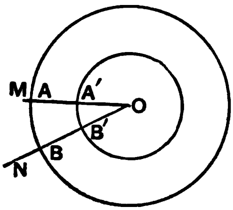
And since ADFC is a parallelogram, AF represents the resultant of the velocities v_1, v_2, v_3.

If the angles at A be right angles, and AF makes angles α, β, γ, with AB, AE, AC, then

$$v^2 = v_1^2 + v_2^2 + v_3^2$$
$$v_1 = v \cos\alpha$$
$$v_2 = v \cos\beta$$
$$v_3 = v \cos\gamma.$$

The same reasoning applies to three accelerations represented by AB, AC, AE; and their resultant will be represented by AF.

25. Angular Velocity.—Let a point P move in a circumference, and let A be the position of the point P at the commencement of the time considered; while the point P describes the arc AB, the radius OA describes the angle AOB. The rotation of the radius marks what is called the angular velocity of the point P. When the radius described equal angles in equal times, *the angular velocity* is uniform, and is equal to the angle described by the radius in a unit of time.



When the radius does not describe equal angles in equal times, the angular velocity at any point is the angle which would be described by the radius if it continued to revolve for a unit of time with a uniform angular velocity.

If the point P moves in a circle with a uniform linear velocity, equal arcs will be described in equal times, and since equal arcs correspond to equal angles at the centre, the angular velocity will also be uniform. The angular velocity is not sufficient to define the motion of a point; for if a point P′ describe the smaller circumference in the same time as the point P describes the larger, the angular velocity will be the same, but the linear velocities will be proportional to the circumferences or the radii. Hence if the radius and angular velocity be given, the linear velocity may be found, and *vice versâ*.

Let r be the radius, v the linear velocity, a the angular velocity.

Let the point describe the circumference c in a time t, then

$$tv = c$$

and $\quad ta = 360°$. Hence $\dfrac{v}{a} = \dfrac{c}{360°} = \dfrac{\pi r}{180°}$.

If a be measured in circular measure, then

$$v = ra \text{ and } a = \frac{v}{r}.$$

Hence the angular velocity is measured by the velocity of a point on the radius at a distance unity from the centre.

Since the angle between tangents at two points is equal to the angle between the radii drawn to these points, angular velocity may be called *rate of change of direction.*

For example, if the angular velocity be 20° per second and the radius 3 feet, we have

$$\frac{v}{20} = \frac{3\pi}{180};$$

$$\text{or } v = \frac{22 \text{ ft.}}{3 \times 7} = 1\tfrac{1}{21} \text{ ft.}$$

When the particle is not describing a circle but a curve MN, its angular velocity about any point O is that of any point A at a constant distance from O in the line joining the particle with O.

EXERCISE II.

1. A body starts in a horizontal direction with a velocity of 24 feet per second and receives a vertical acceleration of 32 feet per second; find its velocity after 4 seconds, and also its distance from the starting point.

2. A body moving with a horizontal velocity of 31 feet per second has a vertical velocity given to it in every second of 32 feet per second; find its velocity after 15 seconds.

3. A body has a velocity which would carry it over 80 feet in 5 seconds in a direction inclined at an angle of 60° to the horizon, and a vertical acceleration which would carry it through 400 feet in the same time; find its velocity at the end of the 5 seconds.

4. A man walking on the deck of a ship moves with a velocity of $4\frac{1}{3}$ feet per second while the ship has a speed of 6 miles an hour; through what distance will the man move in 5 seconds?

5. A boat propelled with uniform velocity of 4 miles an hour at right angles to the direction of a stream which is running at the rate of 3 miles an hour, crosses the stream and arrives at a point half a mile below that which is opposite the starting point; find the time and the distance rowed.

6. A body has a velocity of 60 feet per second towards the north, and a velocity of 6 feet per second is given to it in each of 10 seconds towards the north-east; find the magnitude and direction of its velocity at the end of the 10 seconds.

7. A point has acceleration of 16 and 63 metres per second in two directions at right angles; find the space passed over in 10 seconds.

8. A point has accelerations in three directions at right angles (and therefore not in the same plane) of 49, 168, and 288 feet per second, respectively; find the time required to move over 3033 feet.

9. A particle moves in a circle of radius 40 feet with a velocity of 10 yards per minute; find the angular velocity of the particle.

10. A balloon is carried along by a current of air moving from east to west at the rate of 60 miles an hour, having no motion of its own through the air, and a feather is dropped from the balloon. What sort of a path will it appear to describe as seen by a man in the balloon?

11. A heavy body on a level plane has simultaneously communicated to it an upward vertical velocity of 48 feet per second, and a horizontal velocity of 25 feet per second. Find its greatest height, whole time of motion, and its range.

12. Two particles of masses m and m' set out from the same point O, and travel with uniform velocities v and v' along the straight lines OA and OA′, inclined to one another at an angle a. Prove that the centre of gravity of the two particles (*i.e.* the point in the line joining them which divides this line into parts inversely as the masses) will move along a straight line with an uniform velocity; and determine this velocity in terms of the data of the problem.

CHAPTER III.

FORCE AND MOTION.

SECTION I.—*Falling Bodies.*

26. Introduction.—The readiest and simplest illustrations of the theory of the preceding chapter are to be found in the laws observed by bodies moving in consequence of the earth's attraction. It is proved by experiment that the motion of falling or projected bodies, in a space from which the air has been removed, is uniformly accelerated. We attribute this acceleration and the weights of the bodies to the same cause. Now the cause of motion is termed *force*, and the force is considered uniform or constant when it produces uniform acceleration. Hence we may say that the object of this chapter is to consider a particular case of the motion of bodies acted on by a force which is constant in magnitude and direction.

The chapter is divided into two sections; the first, on falling bodies, illustrates the motion in a straight line of a body under the action of a constant force; the second section, on projectiles, illustrates the resolution and composition of velocities.

27. The Acceleration due to Gravity.—This acceleration is usually denoted by g—, hence

g=the gain of velocity in every second of time;

also g=twice the space passed over in the first second.

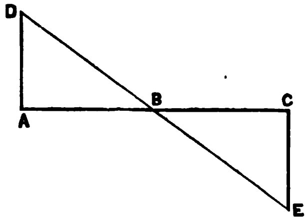
g is not the same for all places, but changes with the latitude of the place and with its elevation above the level of the sea. If l be the latitude of the place, and h its height above the level of the sea, it is found to obey very nearly the following empirical rule :—

$g = 32{\cdot}088\,(1 + {\cdot}00513\sin^2 l - {\cdot}0000003h)$ in British units;

or $g = 9{\cdot}78\,(1 + {\cdot}00513\sin^2 l - {\cdot}0000003h)$ in metrical units.

Hence at the pole $g = 33{\cdot}056$ feet or $9{\cdot}83$ metres per second, and at the equator $g = 32{\cdot}088$ feet or $9{\cdot}78$ metres per second. The average value of g for Great Britain is $32{\cdot}2$ feet per second.

28. Graphic Representation of the Motion of Falling Bodies.—If a body be projected vertically upwards with a velocity v, the motion is graphically represented by the figure. AB represents t the time of ascent, BC the time of descent, AD velocity at starting, EC final velocity, $\triangle$ABD height ascended, $\triangle$BCE height descended. Since the acceleration is the same for an ascending and for a falling body, BD and DE are equally inclined to BA and BC. And since the heights represented by the triangles ABD and CBE are equal, these triangles must be equal in all respects, hence AB$=$BC$=t$ and AD$=$CE$=v$. Also since g is the gain of velocity in one second, and EC or v the gain in t seconds,

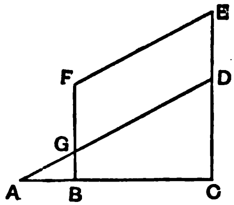


therefore $v = gt$ and s or area ABD$=\frac{1}{2}vt = \frac{1}{2}\frac{v^2}{g}$.

These results may be collected thus :—

When a body is projected vertically upwards with velocity v, it rises for $\frac{v}{g}$ seconds, reaches the height $\frac{v^2}{2g}$, falls in the same time as it took to ascend, and strikes the ground with the velocity v.

Example 5.—A body A is dropped from a certain height; with what velocity must a second body B be projected from the same spot that it may start two seconds after A and overtake A in 10 seconds?



Let AC represent 12 seconds, and AB represent 2 seconds.

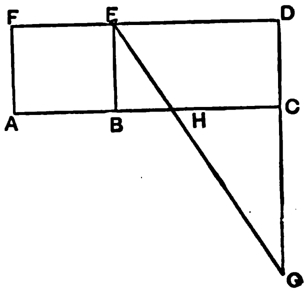
Let CD = the velocity of A after 12 seconds; $\therefore$ $CD = 12g$.

Let BF = the velocity (v) of B at starting. Draw FE parallel to AD and meeting CD in E. Then CE is the velocity of B after 10 seconds. Now the space passed over by A is represented by ACD and that of B by BCEF. But these spaces are the same.

$$\therefore \text{ Area } ACD = \text{area } BCEF$$
$$\therefore \text{ Area } ABG = \text{area } GDEF$$
$$\therefore \tfrac{1}{2} AB \times BG = GF \times BC$$
$$\therefore \tfrac{1}{2} \times 2 \times 2g = (v - 2g) \times 10$$
$$\therefore v = (2{\cdot}2)g.$$

Example 6.—A balloon has been ascending vertically at a uniform rate for 4·5 seconds, and a stone let fall from it reaches the ground in 7 seconds; find the velocity of the balloon, its height when the stone becomes detached, the height reached by the stone, and the time during which it falls.

Take a horizontal line AC to represent the time. Let $AB = 4{\cdot}5$ and $BC = 7$.



Let AF = the velocity v of the stone before it is let fall.

Let CG be the velocity of the stone when it reaches the ground. CG must evidently be drawn in the opposite direction to BE.

Connect the lines as in the figure. Then DG is the whole velocity acquired by a falling body in 7 seconds;

$$\therefore DG = 7g.$$

Area AFEB = space passed over in the ascent by the stone while attached.

Area BEH = space passed over by stone after it becomes detached and before its velocity is zero.

Area GHC = space passed over in the descent.

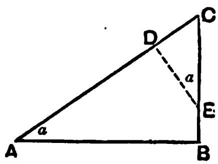
$$\therefore \text{ Area ABHEF} = \text{area GHC} ;$$
$$\text{add area EHCD.}$$
$$\therefore \text{ ACDF} = \text{EGD},$$
$$\text{that is, } 11{\cdot}5v = \tfrac{1}{2} \times 7 \times 7g = 784 ;$$
$$\therefore v = 68 \text{ nearly.}$$

Hence the height at which the stone was detached;

$$\text{or AB} \times \text{BE} = 4{\cdot}5 \times 68 = 306.$$
$$\text{Again, GD} : \text{DE} = \text{GC} : \text{CH},$$
$$\text{or } 224 : 7 = 224 - 68 : \text{CH} ;$$
$$\therefore \text{CH} = 4\tfrac{7}{8}.$$

Hence the stone was falling for $4\frac{7}{8}$ seconds, and therefore rising unattached for $7 - 4\frac{7}{8}$ or $2\frac{1}{8}$ seconds. Its greatest height is represented by $\text{HCG} = \frac{1}{2}\text{HC}.\text{CG} = \frac{1}{2} \times 4\frac{7}{8} \times 156 = 380\frac{1}{4}$ feet.

29. Acceleration Down an Inclined Plane.—We have seen that any acceleration may be resolved into two components in any two directions. Let g, the acceleration of a body falling freely, be resolved along a line DE, making an angle a with the vertical and along a line CA at right angles to DE. The components will be along DE g cos$a$, and along CD, g sin$a$.



(§ 23. vi.) That is to say, a particle falling freely down CB with acceleration g may be supposed to fall down DE with acceleration g cos$a$, while DE moves parallel to itself, so that D moves along CA with acceleration g sin$a$.

If now the acceleration g cos$a$ along DE be neutralised by the introduction of an equal and opposite acceleration, the remaining acceleration will be g sin$a$.

This acceleration is introduced when a particle is left free to move down a smooth rigid plane CA, inclined at an

angle a to the horizon, for such a plane prevents motion perpendicular to itself.

The plane exerts a pressure on the particle perpendicular to itself. Let r be the acceleration at any instant which a force equal to this pressure acting alone would produce in the given particle. Then the components of the acceleration are

(1.) $r-g\cos\alpha$ perpendicular to the plane
(2.) $g\sin\alpha$ down the plane.

The first is zero, and therefore r equals $g\cos\alpha$, and is constant. Hence the pressure on the plane is constant.

If h be the height and l the length of the plane, and f the acceleration down the plane, then since $\sin\alpha=\frac{h}{l}$,

$$\therefore f=g\frac{h}{l}.$$

Thus all the circumstances of the motion of a body down an inclined plane may be found in precisely the same way as in the case of a body falling freely by using $g\frac{h}{l}$ as the acceleration instead of g.

To find the velocity acquired by the body in falling down the length of the plane.

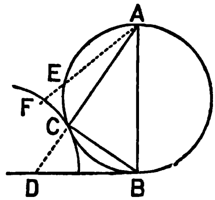
The formula $v^2=2fs$ becomes in this case

$$v^2=2\cdot g\frac{h}{l}\cdot l=2gh.$$

But this is the equation which gives the velocity of a body falling freely down a height $=h$.

Hence the velocity acquired in sliding down a smooth inclined plane is the same as would be acquired in falling freely through a vertical space equal to the height of the plane.

The time of falling from rest down a chord of a vertical circle drawn from the highest point is constant.



Let A be the highest point of a vertical circle, AB a diameter, AC any chord.

Draw the tangent at B meeting AC produced in D. Then the triangles ACB, ABD having an angle A common and the right angles at C and B equal, are similar.

Let t be the time of falling down AC; then

$$AC=\tfrac{1}{2}t^2g\frac{AB}{AD}.$$

$$\text{And } \frac{AC}{AB}=\frac{AB}{AD}\text{; so that } AB=\tfrac{1}{2}t^2g.$$

But if t' be the time of falling down AB; then

$$AB=\tfrac{1}{2}gt'^2$$

That is, t is equal to the time of falling freely down the vertical diameter AB. This establishes the proposition.

In the same manner we may show that the time of falling from rest down a chord passing through the lowest point is constant.

This proposition furnishes a number of problems which are interesting, though, perhaps, without much practical utility, namely, to find what are termed straight *lines of quickest descent* from a point to a curve or from one line to another. For example, suppose a curve touches the circle in the point C, then AC is the line of quickest descent from A to the curve. If any other point F in the curve be joined with A, the line formed will consist of a chord of the circle AE (the time down which is equal to the time down AC) and a part EF without the circle. Hence the time down the whole is greater than the time down AC.

For the solution of such problems we must therefore devise a construction which will furnish a circle having the given point for its highest point, and touching the given line. The following constructions may serve as examples.

(i.) *Find the straight line of quickest descent from a point* A *to a given line* CB, *neither horizontal nor vertical.*

Draw a horizontal line through A, meeting CB in C, and describe a circle touching AC in A and CB in a point D; join AD, and it will be the line required.

(ii.) *From a point* A *without a given circle to the circle.*

Join A with the lowest point L of the circle; then if AL cut the circle in D, AD is the straight line of quickest descent.

(iii.) *From a point* A *within a given circle to the circle.*

Join the point A with the highest point H of the circle, and produce HA to meet the circle again in C; AC is the line required.

Let O be the centre. Join OH, OC, and draw AN parallel to OH, and meeting OC in N, then $\angle$NAC$=\angle$OHC. Therefore $\angle$NAC$=\angle$NCA, hence NA$=$NC, and a circle having N for centre and NA for radius will pass through A and C, having A for its highest point and touching the first circle in the point C.

EXERCISE III.

1. If a body is projected upwards with a velocity of 120 feet in a second, what is the greatest height to which it will rise, and when will it be moving with a velocity of 40 feet per second?

2. The intensity of gravity at the surface of the planet Jupiter

being about 2·6 times as great as it is at the surface of the earth, find approximately the time which a heavy body would occupy in falling from a height of 167 feet to the surface of Jupiter.

3. A man descends the shaft of a mine 4615 feet deep with uniform velocity. Having descended for 10 minutes he drops a stone, which reaches the bottom in 15 seconds; find the velocity with which the man descends.

4. A "lift" ascends with a uniform acceleration of 2 feet per second. After the lift has ascended for 8 seconds a body is dropped from it; in what time and with what velocity will it strike the ground?

5. Through what vertical distance must a heavy body fall from rest in order to acquire a velocity of 161 feet per second? If it continue falling for another second, after having acquired the above velocity, through what distance will it fall in that time?

6. A body thrown vertically upwards strikes the ground after 4 seconds; find the velocity of projection.

7. A body is dropped from a tower, and 2 seconds later a second body is projected after the first, which it overtakes in 10 seconds; find the velocity of projection of the second body.

8. A body is at a given instant moving upward with a given velocity v; show that it will be moving downward with an equal velocity after $\frac{2v}{g}$ seconds, that it will rise through $\frac{v^2}{2g}$ feet, and will reach its highest point after $\frac{v}{g}$ seconds.

9. A body is projected upward with a velocity of 150 feet per second; how high will it have ascended in $6\frac{1}{4}$ seconds?

10. If a body falls freely through 1600 feet, find the velocity it acquires.

11. A rocket having a small piece of lead attached to it, rises vertically with uniform velocity. Seven seconds after starting, and while the rocket is still moving with the initial velocity, the lead becomes detached and reaches the ground in 9 seconds; find the height reached by the lead.

12. A metal ring slides down a smooth rod 9 feet long; divide the rod into three parts so that the time of sliding down each part will be the same.

13. If a yard be the unit of space and a minute the unit of time, find the measure of the acceleration due to gravity.

14. Two particles are let fall from two given heights; find the interval between their starting if they reach the ground at the same time.

15. A stone A is let fall from a certain point, and after it has fallen for a second, another stone B is let fall from a point 100 feet lower down; in how many seconds will A overtake B?

16. A stone A is projected vertically upward with a velocity of 80 feet per second; after 4 seconds another stone B is let fall from the same point; how long will B move before it is overtaken by A, and how far will they then be from the point of projection?

17. In the last example, if only 2 seconds had elapsed, would A ever have overtaken B?

18. The point A is $4g$ feet above B; a body is thrown upward from A with a velocity of $2g$ feet per second, and at the same instant another is thrown upward from B with a velocity of $3g$ feet per second; show that after 4 seconds they will both be at A, moving downward with velocities $2g$ and g feet respectively.

19. A body was thrown from a height $8g$, and struck the ground with a velocity $5g$; what was the velocity of projection?

20. A stone is dropped into a well and is heard to strike the water in 3 seconds; if the velocity of sound $= 35g$, find the depth of the surface of the water.

21. From the top of a room 12 feet high an elastic ball is thrown with just such a velocity that it will rise to the same height if the ratio of the velocity of rebound to the velocity of impact (*i.e.* the co-efficient of elasticity) is $\frac{1}{2}$, find the velocity of projection ($g=32$).

22. If a ball thrown upwards with velocity of 104 feet per second rebounds first at the ceiling, then at the ground, and just rises to the ceiling again; find the height of the room, the co-efficient of elasticity, or the ratio of velocities of rebound and impact being $\frac{1}{4}$ ($g=32$).

23. A body is projected up an inclined plane with a given velocity. Show that the space described in any time is equal to that which would be described in the same time with a uniform velocity equal to half the sum of the velocities at the beginning and end of the time. Hence find the space described on a plane inclined at

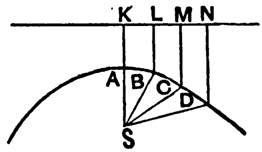
30° to the horizon, while the velocity changes from 48 to 16 feet per second.

24. From a point in a smooth inclined plane a ball is rolled up the plane with a velocity of 16·1 feet per second. How far will it roll before it comes to rest, the inclination of the plane to the horizon being 30°? Also how far will the ball be from the starting-point after 5 seconds from the beginning of motion?

SECTION II.—*Projectiles.*

30. Introduction.—We will now discuss the properties of the path of a projectile, that is to say, of a body thrown in a direction not vertical, and acted on only by its weight, the resistance of the air being neglected. We shall have occasion to refer to the following facts and definitions.

If a fixed straight line and a fixed point be taken, the curve, containing all points which are at the same distance from the line as from the point, is termed a parabola. For example, if SA=AK, SB=BL, SC=CM, SD=DN, etc., the points A, B, C, D, etc., lie on a parabola. The fixed straight line KN is termed the directrix of the parabola, and the fixed point S the focus. A straight line like SK through the focus perpendicular to the directrix is termed the axis, and the point A in which the axis cuts the curve is termed the vertex.



From the above definition of a parabola it is proved in geometry that if a tangent OC be drawn at any point O in a parabola, and then from another point B a straight line BC be drawn parallel to the axis, the ratio $\frac{(OC)^2}{BC}$

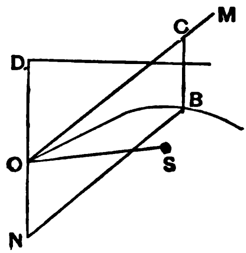
is constant, and equal to four times the distance of the point O from both the focus and the directrix of the parabola, or writing y for OC, x for BC, and a for SO, then $y^2=4ax$. If different points on the parabola be taken as the origin or starting point, we have, of course, equations of the same form, but having different values of the quantity a. If the vertex A be taken as origin, then a is the distance AS or AK, and four times a is termed the *latus rectum*. Hence if the vertex be the origin, and the equation to the parabola be $y^2=lx$, l is the *latus rectum*.

31. Proposition V.—*The path of a projectile is a parabola.*

It is evident that the path will be in the vertical plane through the line of projection: let it lie in the plane of the paper, and let O be the point of projection, and OM the line in which the body is projected; OM will manifestly be a tangent to the curve described.

Let V be the velocity of projection, and C the point at which the body would arrive with this velocity in the time t, so that $OC=Vt$.

From C draw CB vertical, and make $CB=\frac{1}{2}gt^2$, then, if there had been no velocity of projection, the body would have described the space CB in the time t under the action of gravity only; and if gravity had not acted, the body would have been at C; therefore when the body is simultaneously animated by its original velocity V, and that generated by gravity, it will be at B. Let OC $=y$, and $CB=x$,



$$\therefore\ x=\text{CB}=\tfrac{1}{2}gt^2,$$

$$y=\text{OC}=\text{V}t.$$

Eliminate t between these two equations,

$$\therefore y^2 = \frac{2V^2}{g}x$$

$$\text{or, } y^2 = 4ax$$

$$\text{where } a = V^2 \div 2g,$$

and therefore B lies in a parabola, of which the axis is vertical.

From the point O draw OD perpendicular to the directrix and meeting it in D, and let S be the focus, then

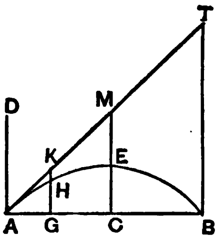
$$V^2 = 2ga = 2g.OS = 2g.OD.$$

Hence the velocity at O, and therefore at any point of the parabola is that which would be acquired in falling from the directrix. The distance of the directrix is therefore independent of the angle of projection. If, then, a number of particles were projected from the point O with the same velocity at the same instant, in different directions, one of which is vertical, all the parabolas described would have a common directrix, which would be just reached by the particle whose path is vertical.

32. Horizontal and Vertical Velocities.—It is generally convenient to resolve the velocity into two components. If, for instance, the velocity at a point be V, and the direction of motion at that point be inclined to the horizon at an angle α, then we may suppose the body to have at the point a horizontal velocity V $\cos\alpha$, and a vertical velocity V $\sin\alpha$. The horizontal velocity is constant throughout the motion, and is therefore always V $\cos\alpha$; the vertical velocity changes, in consequence of gravity. By resolving the velocity in this way, we have the following conclusions :—

(1.) *To find the distance of the focus from the point of projection.*—Since the equation to the path is

$$y^2 = \frac{2V^2}{g}x,$$



the required distance is $\frac{V^2}{2g}$.

(2.) *To find the greatest height.*—It is evidently independent of the horizontal velocity, and is simply due to the vertical velocity. Hence, by § 15

$$\text{greatest height} = \frac{V^2 \sin^2\alpha}{2g}.$$

(3.) *To find the latus rectum of the path.*—The velocity at the vertex is entirely horizontal, and is V $\cos\alpha$; hence, if the vertex be the origin, the equation to the path is

$$y^2 = \frac{2V^2 \cos^2\alpha}{g}x.$$

$$\therefore \text{latus rectum} = 2V^2\cos^2\alpha \div g.$$

(4.) *To find the time of flight.*—This is evidently twice the time taken by gravity to generate the initial vertical velocity,

$$\therefore \text{time of flight} = \frac{2V \sin\alpha}{g}.$$

(5.) *To find the range on a horizontal plane.*—This depends only on the constant horizontal velocity and time of flight, or

$$\therefore \text{Range} = V.\cos\alpha \times 2V \sin\alpha \div g.$$

$$= \frac{V^2 \sin 2\alpha}{g}.$$

(6.) *To find the velocity at any point.*—Let t be the time from the point of projection, then if v be the velocity at time t, and h and k the horizontal and vertical components of v, then

$$h = V \cos\alpha$$
$$k = V \sin\alpha - gt$$
$$v^2 = h^2 + k^2$$
$$\therefore v^2 = V^2 - 2gVt \sin\alpha + g^2t^2.$$

(7.) *To find the equation to the curve referred to rectangular co-ordinates.*—We find the equation when we express the relation between the variable lengths AG, GH, for any point H of the curve.

Let $AG=x$, $GH=y$, $GAK=\alpha$, and $t=$ the time of describing AH.

$$\text{Then } AK=Vt \text{ and } x=AK\cos\alpha=Vt\cos\alpha.$$

$$\text{Also } GK=x\tan\alpha\text{; and } HK=\tfrac{1}{2}gt^2\text{;}$$

$$\text{hence, } y=GK-HK=x\tan\alpha-\tfrac{1}{2}gt^2\text{;}$$

but from the first equation $t=\dfrac{x}{V\cos\alpha}$;

$$\text{hence, the equation is } y=x\tan\alpha-\frac{gx^2}{2V^2\cos^2\alpha}.$$

These equations

i. $x=Vt\cos\alpha$

ii. $y=x\tan\alpha-\tfrac{1}{2}gt^2$

iii. $y=x\tan\alpha-\dfrac{gx^2}{2V^2\cos^2\alpha}$

are frequently useful in solving problems. For example:—

To find the range on an inclined plane.—Let θ be the inclination of the plane and l the range, then in Equation iii. substitute $x=l\cos\theta$, $y=l\sin\theta$.

To find the time of flight on an inclined plane.—Find l the range, and substitute $x=l\cos\theta$ in Equation i.

To find the angle of projection so as to hit a given object.—The height and horizontal distance of the object must be known, and hence they may be substituted for y and x in Equation iii., then, remembering that

$$\frac{1}{\cos^2\alpha}=1+\tan^2\alpha,$$

we shall have a quadratic to determine $\tan\alpha$.

$$y=x\tan\alpha-\frac{g}{2V^2}x^2\,(1+\tan^2\alpha).$$

When the two values of $\tan\alpha$ which answer the equation are real and unequal, there are two directions of projection

which will satisfy the problem; when the two roots are real and equal, these two directions coincide; and when the roots are unreal, the problem is impossible, *i.e.* there is no direction in which the body could be projected with the proposed velocity so as to pass through the given point.

EXERCISE IV.

(Take $g=32$.)

1. A body is projected at an angle a and with velocity V; find the vertical velocity at the time t.

2. A body is projected with velocity of 60 feet per second in a direction making an angle of 30° with the horizon; find its velocity at the end of half a second.

3. In the above example find the range, greatest altitude, and time of flight.

4. If the angle of projection be 45° and greatest height 125 feet, find the velocity of projection, the range, and time of flight.

5. The velocity of projection is 272 feet; find the distance of the focus of the parabola from the point of projection.

6. The velocity of projection is $40\frac{1}{4}$ feet; find the height of the directrix of the parabola.

7. The horizontal range $=1000$ feet, the time of flight $=15$ seconds; find the direction and velocity of projection, and the greatest height.

8. If at the highest point of the path of a projectile the velocity be altered without altering the direction of motion, how will the change affect the time of reaching the horizontal plane which passes through the point of projection?

9. Show that the greatest range up a plane inclined at 30° is two-thirds of the greatest range on a horizontal plane, the initial velocity being the same in the two cases.

10. A body is projected with a velocity V and angle of projection a; determine the velocity with which another must be projected vertically, so that the two may reach the ground at the same instant.

11. A heavy particle is projected from a point with a velocity of

60 feet, and in a direction inclined 30° to the horizon; find its distance from the point of projection at the end of 2 seconds.

12. A number of particles are projected from a fixed point in one plane, so that their least velocity is constant; show that all of them will be found at the same instant on the same vertical line.

13. The horizontal range of a projectile is three times the greatest altitude; find the angle of projection.

14. A body is projected with velocity of 150 feet per second and angle of projection 60°; find the direction, velocity, and height at the end of 5 seconds.

15. The height of a projectile at a distance of 880 yards is 100 feet, and the whole range 1200 yards; find the velocity and direction of projection.

16. Find the range on a plane through the point of projection inclined at an angle of 30°; the initial velocity being 24, and the angle of projection a.

17. In the last example what must be the value of a that the range may be the greatest possible, and what is then the range?

18. If the greatest range up a plane inclined at an angle of 30° is 48 feet, find the velocity of projection.

CHAPTER IV.

FORCE AND MASS.

33. Introduction.—The relation between the amount of substance in a moving body, the acceleration or rate of change of velocity, and the force or the agent which produces change of motion forms the subject of the present chapter. Experiments will be described by which this relation is established, and the relation expressed in the simplest form. We shall show that it is necessary to choose an arbitrary unit in addition to those of length and time, and that it may be either the unit of force or the unit of mass. In the first case the unit of mass, and in the second the unit of force, will be the derived unit. We shall examine the effect of both alternatives, and prove that for the purposes of such exact comparison as is required in physical science, the unit of mass in preference to that of force must be taken as the third fundamental unit.

34. Definitions concerning Matter.—The name *matter* is given to whatever affects the senses. Portions of matter limited in all directions are termed bodies. The quantity of matter in a body is termed its *mass*. When it is required to consider a body so small that its shape and dimensions need not be taken into account, and yet possessing all the properties of matter, such a body is referred to as a *material* *article*.

Matter cannot move spontaneously; whenever a body passes from a state of rest to a state of motion, or from one direction or rate of motion to another, experience shows that the change is always due to some cause exterior to the body.

35. Definition of Force.—*Any cause which changes or tends to change a body's state of rest or motion is termed a force.* It is evident from this definition that the idea of force involves that of mass, but the converse is not necessarily the case. We may imagine an isolated mass of matter placed in such a position that it is not acted on by any external force.

Consider, for example, a mass of iron or brass commonly termed a pound weight. The force attracting it to the earth, or its weight, must not be confounded with its mass or the quantity of matter in it. Each of these things is sometimes called a pound. To avoid ambiguity, we shall refer to the legal material model as *the standard pound weight;* the weight of this body will be called *the weight of a pound*, and any quantity of substance equal to the mass of the standard will be called *the mass of a pound.*

Similar expressions will be used with other so-called standard weights, as, for instance, the grain and the gramme.

36. The Weight of the same Mass varies with Locality.—That the weight of a pound is a variable quantity may be shown experimentally in several ways.

(i.) *The Spring Balance.*—First, we may interpose a force to balance the weight (that is to say, to prevent the body from falling), which is obviously independent of the earth's attraction. For example, if a standard pound weight be suspended from a spring balance furnished with an index and a graduated scale, the weight of the pound will be exactly balanced by the elasticity of the spring, and the degree of tension of the spring will be shown by the position of the index. Now, if the whole apparatus be carried

to different places on the earth's surface, it will be found that the index points to different positions on the scale. Moreover, if it be carried upwards, and allowed to rest at different altitudes, it will be found that the *weight* diminishes as the altitude increases, while it is evident that the *mass* remains the same. Hence we conclude that if it could be carried to a great distance from the earth and all the planets, the index would rise to its initial position, and the weight of the pound would be inappreciable. Here there would be mass without weight.

(ii.) *The Pendulum.*—The range of variation of the weight of a given mass with locality is within 160th of itself, and could not be measured very accurately by so coarse an instrument as the spring-balance. It may, however, be shown with great exactness by pendulum experiments. It will be proved in a subsequent chapter that the squares of the times of vibration of the same pendulum in different places are to one another inversely as the weights of the bob of the pendulum at the same places, or the squares of the numbers of vibrations per hour are directly proportional to the weights of the bob. Now a pendulum may, by careful suspension, be made to vibrate several thousands of times before it stops; consequently it furnishes us with a means of determining with very great exactness the variation of the force of gravity with change of locality.

37. Relation of Mass and Force.—A definite relation exists between the mass of a body, the force acting on it, and the velocity produced in a given time. This relation is a law of nature, and depends for demonstration on experiment. We may express the law in three statements.

(i.) So long as the force acting on a body remains constant, the acceleration is constant.

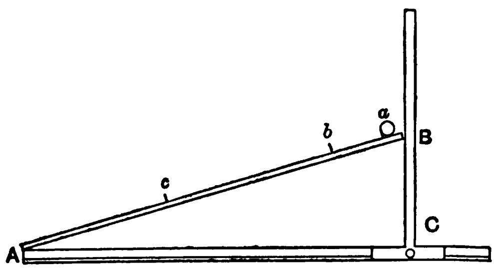
(ii.) If different forces are made to act on the same body, the accelerations are always proportional to the forces.

(iii.) If the same force acts on different bodies, the accelerations are inversely proportional to the masses.

Galileo proved the law by allowing smooth bodies to slide down smooth inclined planes at different elevations. The knowledge which he had of the properties of the inclined plane enabled him very readily to perceive that a body descending on such a plane must be uniformly accelerated, though more slowly than when it falls directly, and is accelerated by its whole weight. By means of the inclined plane, therefore, he was able to bring the whole theory of falling bodies to the test of experiment, and to prove the truth of his original assumption, the uniformity of their accelerations. The following is a modification of the experiment :—

38. Galileo's Experiment.—Let AB be a smooth plane which can be inclined at any angle. Let there be a groove along AB, down which a smooth ivory ball can be made to slide.

Let M be a plate which can be placed at any point in the groove so as to stop the ball at that point. Let *a* mark the position of the front of the ball at starting. By repeated



trials find the points *b*, *c*, *d*, etc., at which M must be placed in order to stop the ball after 1, 2, 3, etc., seconds respectively. Finally, let the distances *ab*, *bc*, *cd*, etc., be measured.

Now it will be found (1) that in any position of the plane the space $bc=3ab$, $cd=5ab$, $de=7ab$, and so on; (2) that as the inclination of the plane is increased, the length ab passed over in a second increases. This relation between the spaces is exactly that which we have proved to arise from a constant acceleration. Hence the experiment proves that while the force producing the motion remains the same, the acceleration is constant. Since the acceleration in each case is equal to twice the length of ab, the second observation shows that as the force producing the motion increases the acceleration increases.

Moreover, further observation proves that the acceleration increases exactly in the same proportion as the force. It is found that if AB be always of the same length, the forces up the plane required to support P in any two positions of the plane are proportional to the height BC. If BC, for instance, be $\frac{1}{10}$th of AB, the force which will support P will be $\frac{1}{10}$th of the weight of P.

Now when the experiments are performed with different heights as CB No. 1 and CB No. 2, it will always be found that

$$ab \text{ No. } 1 : ab \text{ No. } 2 :: \text{CB No. } 1 : \text{CB No. } 2;$$

that is to say, the acceleration is proportional to the force.

39. **Weight Proportional to Mass.**—The effect of variation of mass may be deduced from the fact that different masses falling either vertically or down a smooth plane pass through equal spaces in the same time. A source of error, namely, that arising from the resistance of the air, may be eliminated by making the bodies fall in a tube from which the air has been exhausted.

The necessity for such a precaution is evident. It is a matter of common observation that, under ordinary circumstances, a piece of metal and a feather do not fall with the

same speed; in fact some bodies, as an inflated balloon, for example, instead of falling actually rise. The reason is that when bodies fall under ordinary circumstances, another force opposed to gravity is produced, namely, the resistance of the air on the surface of the body. This resistance is not proportional to the mass of the body, but depends on the surface which the body happens to expose to the air. The feather exposes in proportion to its weight a much greater surface to the air than the piece of metal does, and therefore the air offers a much greater resistance to its descent. In a tube from which the air has been exhausted all bodies fall through the same height in the same time.

The argument from this experiment may be stated as follows:—

If we confine ourselves to portions of substance of the same kind, as for definiteness, pieces of lead, we are able easily to compare the masses as well as the weights; thus, two cubic inches of lead evidently contain twice as much mass as one cubic inch, and half as much mass as four cubic inches. In other words, the masses of any number of pieces are proportional to the volumes. It is found that the weights are also proportional to the volumes; hence the masses are proportional to the weights.

When pieces of different masses fall in vacuo, they pass through the same height in the same time, hence the acceleration is the same for all.

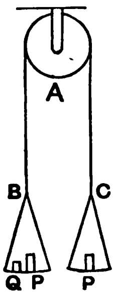
Since it has been shown that increase of weight only would increase acceleration, it follows that increase of mass only must have the opposite effect, for when both increase in the same proportion, the acceleration is not altered, and since the acceleration varies directly as the weight, it must vary inversely as the mass. Thus the relation between force, mass, and acceleration is established, the result being that acceleration varies as the quotient of force by mass.

Again, if bodies of different kinds, as, for example, lead, wood, feathers, be allowed to fall in vacuo, they acquire exactly the same velocity in the same time. Suppose, for definiteness, we have equal weights of each, then since two out of the three things involved in the above relation are the same in all the cases, namely, the weights and the velocities produced in a second, it follows that the masses are equal. Consequently, whatever may be the material composing the bodies, their masses are proportional to their weights.

40. Atwood's Machine.—These very important relations between force, mass, and acceleration admit of a crucial test by means of Atwood's machine.

This machine consists of a pulley, A, capable of turning on its axis without friction, and a fine inextensible thread which passes over the pulley and has its extremities, B and C, attached to weights.

The machine is accompanied by a clock to beat seconds, and a scale to measure distances.



At B and C let there be two scale-pans of equal masses. Let equal masses be put in the pans, and let P denote the sum of one of these masses, and the mass of the scale-pan which contains it: there will then evidently be equilibrium between the two equal weights at B and C, since they tend to move the thread in opposite directions. Now let a small mass Q be added to the scale-pan on the left. The weight of Q will disturb the equilibrium, so that B will descend and C will ascend.

If the weight of Q had to move the mass of Q only, the rate of motion would be too rapid to admit of direct measurement, but it produces motion in itself and also in the larger masses in the two scale-pans. Hence the rate of

motion is slower than that of a falling body, and, indeed, as we shall see, may be made as slow as we please.

Suppose Q, P, P to be made up of parts of equal weight, as, for instance, thin plates of metal, each weighing p. It is then clear that we can alter the weight of Q without changing the whole mass moved; for instance, we may take $2p$ from Q, and place p on each of the equal masses PP, or we may take any (the same) number of times p from each of the equal masses PP, and place both quantities on Q. Again, we may keep the weight of Q unchanged while we increase or decrease the whole mass by adding to or subtracting from PP equal masses. Finally, we may vary at the same time both Q and PP.

Now in whatever way we vary the force which produces the motion (*i.e.* the weight of Q), and the mass moved (*i.e.* the sum of the masses Q, P, P), we find that the acceleration varies directly as the former and inversely as the latter. If F denote the force producing motion, M the mass moved, and f the acceleration, then

$$f \text{ varies as } \frac{\mathrm{F}}{\mathrm{M}}$$

$$\text{or F varies as } f\mathrm{M}.$$

From this it follows by a well-known algebraical rule of variation, that F must *equal* fM multiplied by some quantity c, which remains constant so long as we preserve the same units of measurement, but which depends on these units.

Now for the sake of simplicity we so arrange the units that this constant may be unity, and we write :—

$$\mathrm{F} = f.\mathrm{M}. \quad . \quad . \quad . \quad . \quad (a).$$

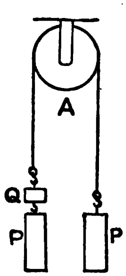
41. Relation between Units of Force and Mass.—The substitution of equality for variation, however, restricts us in the choice of units. There are four units involved in the equation, since f involves two, namely, the units of length and time. It is evident that if we select arbitrarily three

of the four units, the fourth is settled by the equation. For example, if we select our units of length, time, and mass, the force, which, when acting on the unit of mass will produce a unit of velocity in a unit of time, must be the unit of force, for otherwise when the product on the right-hand side of the above equation (a) becomes 1, or unity, the quantity on the left would not be unity. If we select arbitrary units of length, time, and force, the unit of mass must be derived from them by means of the equation (a). Hence, by writing the relation in this simple form we are bound to choose our units so that *the unit of force when acting on a unit of mass shall produce a unit of velocity in a unit of time.*

Let the foot be the unit of length and the second the unit of time, and let us then inquire what will be the relation between the units of mass and force.

The unit of acceleration is the velocity of 1 foot per second gained in a second. It is that acceleration which would cause a body moving uniformly with it, to pass over 6 inches in the first second of its motion.

Let us arrange a force and mass so as to secure this acceleration. For example, take any mass Q+2P, and arrange it as in the figure. Let the masses Q, P, and P be so adjusted that P+Q descends through 6 inches in the first second of motion. The force producing the motion is the weight of Q, the mass moved is Q+2P, and the acceleration is unity.



Let g be the acceleration of a body falling freely; then if the weight of Q moved the mass of Q only, the acceleration would be g. But in order that while the weight of Q is the force, the acceleration may be reduced from g to 1, the mass moved must be increased from Q to gQ. Hence we have

the following important result:—*When in the above arrangement the acceleration is unity, the whole mass moved is* g *times the mass whose weight produces the motion.*

If now we choose the weight of Q for the unit of force, the mass Q+2P will be the unit of mass, and *vice versâ.*

Hence, if the unit of acceleration be a foot per second gained in a second, the unit of mass will be 32·2 times that mass whose weight is the unit of force. If one unit is fixed the other is fixed.

42. Gravitation Unit of Force.—Let Q be the mass of a pound, and let its weight be the unit of force, then $Q+2P(=gQ)$ is the unit of mass. That is to say, if the weight of a pound be the unit of force, the mass of g pounds is the unit of mass. Similarly, if the weight of a gramme be the unit of force, the mass of g grammes is the unit of mass. These are termed gravitation units.

Example 7.—If the number 10 is taken as the measure of the weight of 10 pounds, what is the measure of the mass of 10 pounds?

Ans. g lbs. of mass $=1$;

$$\therefore \text{ 1 lb. of mass} = \frac{1}{g};$$

$$\therefore \text{ 10 lbs. of mass} = \frac{10}{g}.$$

Example 8.—What is the acceleration produced by a force equal to the weight of 5 pounds on a mass of 20 pounds?

Ans.—The force being taken as 5, the mass must be represented by the number $20 \div g$.

$$\therefore \text{ since } F = fM$$

$$5 = f \times \frac{20}{g};$$

$$\therefore f = \frac{g}{4} = 8 \text{ nearly.}$$

Example 9.—If P be the measure of the force in gravitation

units, and W the weight of the mass upon which it acts, find the acceleration.

The force being P, the mass $=\frac{W}{g}$; therefore

$$P=f\times\frac{W}{g}, \text{ or } f=g\frac{P}{W}.$$

Example 10.—If a force equal to a weight of 40 kilogrammes moves a body which weighs 100 kilogrammes, find the acceleration.

If the weight of the body moved its mass, the acceleration would be 32·2; hence the question may be put as one of proportion. If a force of 100 produces an acceleration of 32·2, what acceleration will a force of 40 produce?

$$\text{As } 100 : 40 :: 32{\cdot}2 : x;$$
$$\therefore x=12{\cdot}88.$$

Example 11.—If a body weighing 1000 pounds lie on a platform which is descending with an acceleration of 8·05 feet per second, find the pressure on the platform.

If the platform were at rest it would prevent an acceleration of 32·2 feet per second, and would bear a pressure of 1000 pounds. But as it descends with acceleration 8·05, it prevents acceleration of 24·15 only. Hence

$$\text{as } 32{\cdot}2 : 24{\cdot}15 :: 1000 : x;$$
$$\therefore x=750 \text{ lbs.}$$

Example 12.—A weight of 60 pounds is suspended by a string from a point ascending with a velocity which increases uniformly at the rate of 10 feet per second; find the tension of the string.

A tension of 60 pounds prevents an acceleration of 32·2; what is the tension which will prevent an acceleration of 32·2 and also produce an acceleration of 10 in the opposite direction?

$$\text{as } 32{\cdot}2 : 42{\cdot}2 :: 60 \text{ lbs.} : x;$$
$$\therefore x=78{\cdot}6 \text{ lbs.}$$

43. Variation of Gravitation Unit of Force with Locality.—It has already been said that the weight of a pound varies according to locality. It would not generally be sufficient, therefore, to say simply that the unit of force shall be the weight of a pound without saying at what place on the earth's surface the weight is to be taken. If, how-

ever, it is required only to compare two or more forces in the same place with one another, then since the weight of a pound in the same place is always the same, this weight will be a convenient unit for the comparison, and no error whatever will result from the choice of the unit. It is particularly convenient, also, when the strength of a structure required to support a given mass is to be found. If the weight of the mass change, so will the force necessary to support it, and the weight of a given mass, although variable, is here exactly what is required for the unit.

A variable unit would, however, be exceedingly inconvenient in nearly all cases of scientific investigation. For instance, to compare the earth's horizontal magnetic force at different places, the same standard must be employed, and this cannot therefore be the weights of a given mass at the respective places. It is true we might transfer the unit of weight from one place to another by finding an equivalent force of a kind which, unlike weight, does not change with locality, as, for example, the elastic force of a spring-balance. But, besides the clumsiness of such a plan, there is a further objection, namely, that the balance could only be used for the measurement of forces, and not for the comparison of masses at the different places. Being graduated at one place, it would give different indications when the same mass was weighed by it in different places, and as balances are used far more frequently to measure masses than forces, this mode of avoiding a variable unit of force would be likely to lead to a variable unit of *mass*.

There is, however, a more satisfactory mode of meeting the requirements of science in this respect, and this we will next proceed to explain.

44. The Absolute Unit of Force.—Let us now choose the unit of mass first, and then derive the unit of force from

it, keeping the foot and the second as the units of length and time.

Let the whole mass $2P+Q$ (*see* fig. at page 54) be the mass of a pound, and, by removing mass from P to Q or Q to P, without altering the total amount, let it be so arranged that the acceleration shall be unity. Then we shall have as before, $gQ=2P+Q=1$ pound. That is to say, the mass of Q is the mass of a pound divided by g or 32·2, and therefore the unit of force, or the weight of Q, is the weight of a pound divided by g. Hence, there will be g units of force in the weight of a pound.

This unit of force is an invariable quantity; for though it is true that the weight of a pound varies with the locality, so does g, the acceleration due to gravity, and in exactly the same proportion; hence *if the weight of the unit of mass at any place be divided by the value of* g *at the same place, the result is a constant quantity, which is termed the absolute unit of force.*[1]

[1] "The word 'absolute' in the present sense is used as opposed to the word 'relative,' and by no means implies that the measurement is accurately made, or that the unit employed is of perfect construction; in other words, it does not mean that the measurements or units are absolutely correct, but only that the measurement, instead of being a simple comparison with an arbitrary quantity of the same kind as that measured, is made by reference to certain fundamental units of another kind treated as postulates. An example will make this clearer:—When the power exerted by an engine is expressed as equal to the power of so many horses, the measurement is not what is called absolute; it is simply the comparison of one power with another arbitrarily selected, without reference to units of space, mass, or time, although these ideas are necessarily involved in any idea of work. Nor would this measurement be at all more absolute if some particular horse could be found who was always in exactly the same condition and could do exactly the same quantity of work in an hour at all times. The foot-pound, on the other hand, is one derived unit of work, and the power of an engine when expressed in foot-pounds is measured in a kind of absolute measurement, *i.e.* not by reference to another source of power, such as a horse or a man, but by reference to the units of weight and length simply, which have been long in general use and may

For example, if the unit of mass is the mass of an ounce, the units of length and time being the foot and the second, the unit of force is the weight of an ounce divided by 32·2, or the weight of an ounce must be represented by the number 32·2, and the weight of p ounces by 32·2 p. If the units of length, time, and mass be the metre, second, and the mass of a gramme, since g is 9·8 metres per second, the

be treated as fundamental. In this illustration, chosen for its simplicity, the unit of force is assumed as fundamental and as equal to that exerted by gravitation on the unit of mass, but this force is itself arbitrarily chosen and is inconstant, depending on the latitude of the place of the experiment.

"In true absolute measurement the unit of force is defined as the force capable of producing the unit of velocity in the unit of mass when it has acted on it for the unit of time. Still simpler examples of absolute and non-absolute measurements may be taken from the standards of capacity. The gallon is an arbitrary or non-absolute unit. The cubic foot and the litre or cubic decimetre are absolute units. In fine, the word 'absolute' is intended to convey the idea that the natural connection between one kind of magnitude and another has been attended to, and that all the units form part of a coherent system. It appears probable that the name of 'derived units' would more readily convey the required idea than the word 'absolute,' or the name of 'mechanical units' might have been adopted; but when a word has once been generally accepted, it is undesirable to introduce a new word to express the same idea. The object or use of the absolute system of units may be expressed by saying that it avoids useless co-efficients in passing from one measurement to another. Thus, in calculating the contents of a tank, if the dimensions are in feet, the cubic contents are given in cubic feet without the introduction of any co-efficient or divisor, but to obtain the contents in gallons the divisor 6·25 is required. If the power of an engine is to be deduced from the pressure on the piston and its speed, it is given in foot-pounds or metre-kilogrammes per second by a simple multiplication; to obtain it in horse-power, the co-efficients 33,000 or 550 must be used. No doubt all the natural relations between the various magnitudes to be measured may be expressed and made use of, however arbitrary and incoherent the units may be. Nevertheless, the introduction of the numerous factors then required in every calculation is a very serious annoyance; and, moreover, where the relations between various kinds of measurement are not immediately apparent, the use of the coherent or absolute system will lead much more rapidly to a general knowledge of these relations than the mere publication of formulæ."—*British Association Report*, 1854.

unit of force will be the weight of one gramme, divided by 9·8, or there will be 9·8 units of force in the weight of a gramme.

45. Units of Mass.—The British legal standard of mass is a piece of platinum, deposited in the office of the Exchequer, and termed the Imperial Standard Pound Avoirdupois. The 7000th part of this is termed the *grain.*

The French standard of mass is the Kilogramme des Archives, also made of platinum. The 1000th part of the kilogramme is the gramme, which is intended to be the mass of a cubic centimetre of distilled water at a temperature of 4° C. The kilogramme is equal to 15,432·34874 grains.

The French method of obtaining the unit of mass is therefore to name a standard substance, and then to derive the unit of mass from the unit of length by means of the standard substance. If we apply the same principle to the British unit of length, taking water as the standard substance, since a cubic foot of water weighs nearly 1000 ounces or $62\frac{1}{2}$ pounds, we shall obtain this mass (1000 ounces nearly) as the unit of mass, and the weight of the mass divided by 32·2 as the unit of force.

If the pound be chosen as the unit of mass, we may assume an imaginary substance such that 1 pound of it occupies a cubic foot. Such a substance would evidently be $62\frac{1}{2}$ times lighter than water; there would be $62\frac{1}{2}$ units of mass in a unit of volume, or in other words, the *density* of water, compared with the imaginary substance, would be $62\frac{1}{2}$.

46. Definition of Density.—The density of a body is the number of units of mass in a unit of volume of the substance.

For example, if the foot and the pound be the units of length and mass, since a cubic foot of iron weighs 432 pounds, the density of iron will be 432. If the metre and the kilogramme be respectively the units of length and mass, since

a cubic metre of distilled water at 4° C weighs 1000 kilogrammes, the density of water will be 1000.

Note.—In the following eight examples, the unit of length is the foot, and the unit of time the second. It must be noticed that if we have a mass and a force in the same example with the same designation, as for instance, a mass of p grammes and a force equal to the weight of q grammes, we may proceed in two ways—we may say that the measure of the mass is p, and the absolute measure of the force $32 \cdot 2\, q$, or that the measure of the force is q, and the gravitation measure of the mass $p \div 32 \cdot 2$.

Example 13.—What is the acceleration produced by a force equal to the weight of 5 pounds on a mass of 20 pounds?

Take the mass of a pound as the unit of mass, then the absolute measure of the weight of 5 pounds is $5g$.

Let f = the acceleration,

$$\therefore 5g = f \times 20$$

$$f = \frac{g}{4} = 8 \text{ nearly.}$$

Example 14.—Weights of 17 and 15 ounces respectively are connected by a flexible and inextensible string which passes over a smooth pulley; find the acceleration.

Let an ounce be the unit of mass; then the whole mass moved is 32 units.

The force in this case is the weight of 2 ounces, the absolute measure of which will be $2g$.

Let f be the acceleration.

$$\therefore 2g = f \times 32;$$

$$\therefore f = 2 \text{ nearly.}$$

Example 15.—The two scale-pans in an Atwood's machine each weigh two ounces. Divide $76\frac{1}{2}$ ounces into two parts P and Q, so that when placed one in each pan they may move through 9 feet in 3 seconds, supposing the acceleration due to gravity to be $32 \cdot 2$.

The space passed through in 3 seconds is 9 times that in the first second; hence the weight is to descend 1 foot in the first second, and consequently the acceleration is to be 2 feet per second gained in a second.

Let the ounce be the unit of mass; then the whole mass moved will be $80\frac{1}{2}$.

Let D ounces be the difference of the two masses P and Q, then $g\text{D}$ = the absolute measure of its weight, therefore

$$g\text{D}=2\times 80\tfrac{1}{2}$$

$$\therefore \text{D}=5 \text{ ounces.}$$

Hence $76\frac{1}{2}-5$, or $71\frac{1}{2}$, is twice the smaller mass, Q.

$$\therefore \text{Q}=35\tfrac{3}{4} \text{ ounces and } \text{P}=40\tfrac{3}{4} \text{ ounces.}$$

Example 16.—If the weights on an Atwood's machine be P and Q, find the tension of the string.

Let t be the tension of the string in absolute units.

Let P and Q represent the masses moved, and let P be the greater.

Let f be the acceleration of P downwards, and of Q upwards.

The force producing the motion of P is the excess of its weight over t, and its absolute measure is therefore $g\text{P}-t$.

Similarly the absolute measure of the force producing the motion of Q is $t-g\text{Q}$.

From the motion of P we have $g\text{P}-t=f.\text{P}$.

From the motion of Q we have $t-g\text{Q}=f.\text{Q}$.

Multiply the first equation by Q, the second by P, and subtract;

$$\therefore 2g\text{PQ}=t\ (\text{P}+\text{Q}).$$

$$\therefore t=g.\frac{2\text{PQ}}{\text{P}+\text{Q}} \text{ absolute units.}$$

If T be the gravitation measure of the tension $t=g\text{T}$, and therefore

$$\text{T}=\frac{2\text{PQ}}{\text{P}+\text{Q}}.$$

Example 17.—A body weighing W pounds lying on a smooth horizontal table is attached, by means of a thread passing over a pulley on the edge of the table, to a weight of P pounds. Find the acceleration.

The mass moved $=\text{W}+\text{P}$

The absolute measure of the force $=g\text{P}$;

$$\therefore g\text{P}=f.\ (\text{P}+\text{W}),$$

$$\text{or } f=\frac{g\text{P}}{\text{P}+\text{W}}.$$

Example 18.—In the above example, if the table be rough, and

the resistance be m times the weight of W (m being a small fraction), find f.

The force is the excess of the weight of P over the resistance, hence its absolute measure is g (P − mW).

$$\therefore g\,(P - mW) = f\,(P + W),$$

$$\text{or } f = \frac{g(P - mW)}{P + W}.$$

Example 19.—A body W, weighing 15 pounds, is moved up a smooth inclined plane over 27 inches in 3 seconds by a body P which falls vertically. Having given that there would be rest if P were 2 pounds, find P, supposing g to be 32.

Let f = the acceleration.

Since 27 inches or $2\frac{1}{4}$ feet $= \frac{1}{2} f \times 3^2$

$$\therefore f = \tfrac{1}{2}.$$

If the unit of mass be a pound, the whole mass will be P + 15.

The force is the excess of P over the force which would just support W; hence,

$$\text{its absolute measure} = g(P - 2)$$

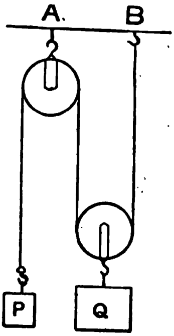
$$\therefore g(P - 2) = \tfrac{1}{2}(P + 15)$$

$$64\,P - 128 = P + 15$$

$$63\,P = 143$$

$$\therefore P = 2\frac{17}{63}\text{lbs.}$$

Example 20.—A weight Q suspended from a moveable pulley is lifted by a smaller weight P attached to one end of a flexible thread the other end of which is attached to a beam. If P weighs 12 ounces and Q and the pulley together weigh 20 ounces, find the acceleration and the tension of the thread, on the suppositions that there is no friction, that the parts of the thread are parallel, and that g = 32.



Let f = the acceleration of Q upwards.

Since it is evident that if Q ascends through any space whatever, P descends through twice this space, the acceleration of P downwards is $2f$.

Let t be the tension of the string in absolute measure.

The force which moves $Q=2t-20g$.
The force which moves $P=12g-t$.

$$\text{Hence } 2t-20g=f\times 20$$
$$\text{and } 12g-t=2f\times 12.$$

Multiplying the second equation by 2, and adding,

$$\therefore 4g=68f,$$
$$\therefore f=\frac{32}{17}.$$

Substituting for f,

$$\therefore \text{the tension}=g\cdot\frac{12\times 15}{17} \text{ absolute units.}$$
$$=\frac{12\times 15}{17} \text{ oz. gravitation units.}$$

47. Recapitulation of the Relation between Units.

(i.) Let V be the volume of a cube whose side is l linear units,

$$\therefore V=l^3,$$

and the unit of volume must be the volume of a cube whose side is the unit of length.

(ii.) Let a particle moving with uniform velocity v pass over l units of length in time t,

$$\therefore l=vt.$$

Hence $v=1$ when $l=1$ and $t=1$; or the unit of velocity is that of a particle which passes over a unit of length in a unit of time.

(iii.) Let v be the velocity produced in a particle in t units of time when the motion is uniformly accelerated by a velocity a in every unit;

$$\therefore v=at,$$

therefore $a=1$ when $v=1$ and $t=1$, or a unit of acceleration is that which produces a unit of velocity in a unit of time.

(ii.) and (iii.) combined give us

$$l=at^2.$$

(iv.) If V be the volume of a body, m its mass, and d its density, then

$$m=Vd,$$

hence $m=1$ when $V=1$ and $d=1$, or the unit of mass is the mass of a unit of volume of the standard substance.

(v.) If a force F act on a mass m, and if a be the resulting acceleration,

$$F=am;$$

hence $F=1$ when $a=1$ and $m=1$; or the unit of force is that which produces unit of acceleration in a unit of time.

A particular case of (v.) is when the force is the weight w of the mass m; then

$$w=gm;$$

hence, when $m=1$, $w=g$, and therefore the unit of weight is the g-th part of the weight of a unit of mass.

Now we have here five equations and eight quantities, hence only three of these quantities are independent; that is to say, if three of them be given, all the others can be found.

Again, if three of the units of a system B are known in terms of the corresponding units of another system A, then all the units of B can be determined in terms of those of A by the above formulæ. For consider any one of them, as for instance the third, $v=at$. The units of the system A satisfy the equation; and besides, if an acceleration a' times this unit give a velocity v' times the unit of velocity in t' units of time, then

$$v'=a't'.$$

Hence, if the units of velocity and time in system B be respectively v' and t' times those of A, the unit of acceleration in B will be a' times that in A, if a' satisfies the equation $v'=a't'$.

We have, therefore, the following rule :—

Rule 1. Write down the above five formulæ with dashed letters, and let these stand for the measures of the new units in terms of the old; substitute the numerical values of those that are known, and solve the equations for the rest.

Or, since the measures of the same magnitude in the two systems of units are inversely as the units, it is clear that we can make the dashed letters stand for the ratios of the measures of fixed quantities in the two systems.

Example 21.—If a yard be the unit of length, and the acceleration of gravity be represented by 96·6, what must be the unit of time?

When the unit of length is the foot, the acceleration of gravity is 32·2, but in the given system the same acceleration is 96·6, or

three times 32·2, and hence the unit is one-third of that when the foot and the second are units.

Again, the new unit of length is three times the foot, and therefore if t' seconds be the new unit of time,

$$v'=a't',$$
$$\text{and } l'=v't';$$
$$\therefore l'=a't'^2,$$
$$\text{or } 3=\tfrac{1}{3}t'^2;$$
$$\therefore t'=3.$$

Example 22.—If mercury be the standard substance, and if the inch and second be the units of length and time, find the units of force and mass, having given that the density of mercury is 13·6 times that of water, and that a cubic foot of water weighs 1000 ounces.

If we take water as the standard substance, and the foot and the second as the units of length and time, we have the unit of mass 1000 ounces and the unit of force the weight of 1000 ounces ÷ 32·2.

$$\text{From formula (i.) } V'=l'^3=\frac{1}{1728}.$$

$$\text{From formulæ (ii.) and (iii.) } a'=\frac{l'}{t'^2}=\frac{1}{12}.$$

$$\text{From formula (iv.) } m'=V'd;\ \therefore m'=\frac{13·6}{1728}.$$

$$\text{From formula (v.) } F'=a'm'=\frac{13·6}{12^4}.$$

Consequently the new unit of mass is the mass of $\frac{13·6}{1728}$ times 1000 oz., and the new unit of force is the weight of

$$\frac{13·6}{20{,}736} \text{ times } \frac{1000}{32·2} \text{ oz., or } \frac{68{,}000}{3{,}338{,}496} \text{ oz.}$$

The process of changing the units above described may be put into a different form by means of the *dimensions* of the quantities considered.

Definition of Dimensions.—An expression involving the three symbols l, t, m in the same degree as the units of length, time, and mass are involved in the measure of a given physical quantity, is said to have the same *dimensions* as the given quantity.

For example :—

Let l, t, m stand respectively for the units of length, time, and mass, and let a, b, and c represent numbers.

(i.) Then since an area = length × breadth

$$= al \times bl = abl^2;$$

$\therefore$ the dimensions of an area are those of l^2.

(ii.) Since a volume = length × breadth × thickness

$$= al \times bl \times cl = abcl^3;$$

$\therefore$ the dimensions of a volume are those of l^3.

(iii.) Since a plane angle = arc ÷ radius

$$= al \div bl = a \div b,$$

the dimensions of a plane angle are zero.

(iv.) A velocity = distance ÷ time

$$= al \div bt,$$

hence the dimensions of velocity are those of $\frac{l}{t}$.

(v.) An acceleration = a velocity ÷ time

$$= \frac{al}{bt} \div ct$$

$$= \frac{a}{bc} \cdot \frac{l}{t^2}.$$

Hence the dimensions of acceleration are those of $\frac{l}{t^2}$.

(vi.) Since a force = an acceleration × mass, the dimensions of force are those of $\frac{lm}{t^2}$.

(vii.) By definition on page 60, the density of a body is its mass ÷ its volume. Hence the dimensions of density are those of $\frac{m}{l^3}$.

Rule II.—To compare the units in two different systems of the same kind of quantity (which we will denote by Q), find the dimensions of the unit of Q, and substitute for l, t, and m the numbers of times the units of length, time, and mass in the second system contain the corresponding units in the first system ; the result will be the number of times the unit of Q in the second system contains the corresponding unit of the first system.

Example.—Compare the unit of acceleration based on the yard and the minute with the unit of acceleration based on the foot and second.

In the dimensions of acceleration $=\frac{l}{t^2}$ for l write 3, and for t write 60.

The result $\frac{3}{3600}$ is the number of times the yard-minute unit of acceleration contains the foot-second unit of acceleration.

Example.—The same question as in example 22.

Take as the initial system the foot-second-ounce system, then the unit of force is the weight of $\frac{1}{32 \cdot 2}$ oz., and the unit of density that of a substance, a cubic foot of which weighs an ounce.

Hence, to form a new system the multipliers are—

$$l=\frac{1}{12},\ t=1,\ m \text{ unknown},\ \mathrm{F}=\frac{lm}{t^2}=\frac{m}{12},\ \rho=\frac{m}{l^3}=12^3m.$$

But in the second system the unit of density is $1000 \times 13 \cdot 6$ times the corresponding unit of the first.

$$\therefore\ 12^3m=13600\ ;$$

$$\therefore\ m=\frac{13600}{12^3}\ ;$$

$$\therefore\ \mathrm{F}=\frac{13600}{12^4}.$$

Hence the new unit of mass is $\frac{13600}{12^3}$ oz. and the new unit of force the weight of $\frac{13600}{12^4} \div 32 \cdot 2$ oz.

A committee of the British Association recommend that for scientific purposes the system of units shall be based on the Centimetre, Gramme, and Second, and termed the C. G. S. system.

EXERCISE V.

1. If the number 20 be taken as the measure of the weight of 20 kilogrammes, what is the measure of the mass of 20 kilogrammes, the acceleration of gravity being 9·8 metres per second?

2. What is the acceleration produced by a force equal to the weight of 100 pounds on a mass of 1000 pounds.

3. If a body weighing 1000 pounds be suspended by means of a cord from a support A, find the tension of the cord,

(i.) When A is ascending with velocity which increases in a second at the rate of 8 feet per second.

(ii.) When A is descending with a velocity which is retarded in a second at the rate of 12 feet per second (the acceleration of gravity being taken as 32).

4. How far will a force equal to the weight of 10 pounds move a mass of 1000 pounds in 4 seconds?

5. What mass will be lifted by a force equal to the weight of 100 ounces so as to produce a velocity of 100 feet per second in 4 seconds?

6. What force will lift a mass of 1000 ounces vertically upwards to a height of 80 feet in 10 seconds, and what will be the velocity of the mass at the end of the time?

7. How far will a weight of 50 pounds be moved by a pressure of 1 pound in 10 seconds?

8. What pressure, by acting for 6 seconds, will produce a velocity of 120 feet per second in a weight of 12 pounds?

9. If weights of 3 and 5 pounds are connected by a string which passes over a fixed pulley, what will be the velocity generated in a second of time?

10. In the above example, how far will the heavier weight descend in 5 seconds?

11. If the weights be respectively 15 pounds and 17 pounds, how long will they take to move through 144 feet?

12. A weight of 2 pounds hangs over the edge of a smooth table, and draws a weight of 50 pounds laid on the table; what velocity will be acquired in $2\frac{1}{2}$ seconds?

13. How far will either of the weights move in 5 seconds?

14. A mass originally at rest is acted on by a force which in 1-368th of a second gives to it a velocity of $5\frac{1}{4}$ inches per second; show what proportion the force bears to the weight of the mass.

15. If a particle moves in consequence of the continued action upon it of a constant force, show what is the character of the resulting motion, and in what manner it depends on the magni-

tude of the force and the mass of the particle. As a special case, show how the resulting motion would be changed if the mass of the particle were trebled and the intensity of the force acting upon it were doubled.

16. If the mass of a pound be the unit of mass, and 1 foot and 1 second be the units of space and time, how would you define the unit of force, and how many such units of force are there in the weight of $1\frac{1}{2}$ pounds?

17. A steam-engine moves a train weighing 60 tons, including engine, on a level road from rest, and acquires a speed of 5 miles an hour in 5 minutes. If the same engine move another train, and give it a speed of 7 miles an hour in 10 minutes, find the weight of the second train, supposing the resistance to amount to the same in both cases.

18. The two scale-pans in an Atwood's machine each weigh 1 ounce; divide 118·75 ounces into parts A and B, so that when placed one in each pan they may move through 16 feet in 4 seconds, the acceleration of gravity being 32·2.

19. What velocity will a weight of 10 pounds falling vertically give to a weight of 60 pounds, which it draws along a horizontal smooth table in 4 seconds?

20. With a single moveable pulley 5 pounds draw up 8 pounds; find how long it will take the 8 pounds to ascend 10 feet?

21. If the measure of a certain force in British absolute units (foot-second-pound) be 56, what will be the measure, when 1 cwt. is the unit of force, a yard the unit of length, and a minute the unit of time?

22. If the yard be the unit of length, and the acceleration of gravity be represented by 2415, find the unit of time.[1]

23. If the unit of weight be an ounce, the unit of time a second, and a cubic foot of the standard substance weigh 162 pounds, what must be the unit of length?

24. If a feet and m seconds are units, the measure of an acceleration is f; find the measure when a' feet, and m' seconds are units.

[1] Generally 32 is taken as an approximation to the acceleration of gravity in ordinary units, but in some questions like this one, taken from examination papers, 32·2 must be used.

25. If the area of a field of 10 acres be represented by 100, and the acceleration of gravity be $58\frac{2}{3}$, find the unit of time.

26. If 8 feet per second be the unit of velocity, the acceleration of gravity the unit of acceleration, and a ton the unit of mass, find the density of the standard substance.

27. If the number of seconds in the unit of time be equal to the number of feet in the unit of length (a cubic foot of the standard substance weighing 13,500 ounces, and the unit of weight being 750 pounds), find the unit of time.

28. When an inch is the unit of length, and t the unit of time, the measure of a certain acceleration is a: when 5 feet and 1 minute are the units of length and time respectively, the measure of the same acceleration is $10a$; find t.

CHAPTER V.

MOMENTUM AND ENERGY.

47. Introduction.—The object of the present chapter is to discuss the relations, differences, and uses of two quantities involving the mass and velocity of a moving body. One involves the first power of the velocity, and is termed *momentum*, and the other involves the square of the velocity, and is termed the *energy* of the body. It will be shown that force and change of momentum are connected as cause and effect, the action of a force for any time producing change of momentum proportional to the force and time, and that the time required by any force to bring a moving body to rest is proportional to the momentum of the body; hence momentum has sometimes been called *accumulated force.* When we consider the space through which the body moves while the force is acting on it, we find that the action of a force through any space produces change of energy proportional to the force and space, and that the space through which a body will move against any force before it is brought to rest is proportional to the energy of the body. The term accumulated force being appropriated to momentum, and the product of force and distance being termed work, the energy of a moving body may be termed its *accumulated work.* Both the momentum and the energy of

a moving body have in time past been termed the *force of the moving body*, and the confusion arising from this ambiguity led to a remarkable controversy towards the end of the seventeenth century. The force of a moving body was said to be measured by the effect which might be produced by the body while it was being brought to rest, but a very little consideration will show that there are two effects produced by a moving body, one of which is proportional to the square of the velocity, while another is proportional to the velocity simply. For instance, suppose three bodies, A, B, and C, are projected with different velocities, and are brought to rest in distances and times as follows, their motions being retarded by equal resistances:—A in 100 yards and 5 minutes, B in 160 yards and 10 minutes, C in 200 yards and 12 minutes, shall we say that the force of the moving body B is twice that of A since B moves twice as long as A, or that the force of C is twice that of A, for C moves twice as far as A? This question we proceed to answer. Suppose a body to be projected vertically upwards with any velocity. The force which acts upon it bringing it to rest, and then changing the direction of its motion, is the force of gravity, and to determine how much of this force is required to reduce the body to rest we may rather inquire into the retardation which that force produces in a given *time*, or while the body is moving over a given *space*. In other words, we may either inquire how long the motion will continue, or how far it will carry the body before it is entirely exhausted. The length of the time that the uniform resistance must act before it reduces the body to rest is known (§ 28) to be proportional to the velocity of projection, but the length of the line which the moving body describes, while subjected to this uniform resistance, is known to be proportional to the square of the velocity. It is necessary to avoid ambiguity that these two effects of the motion

of a body should be called by different names; the first is the momentum, and the second the energy of the body. It is advisable not to use the phrase "force of a moving body" at all, but if used it must be equivalent to the momentum.

SECTION I.—*Momentum.*

48. When a uniform force acts on a body for a unit of time, the force is measured by the velocity produced, multiplied by the mass of the body, or using our previous notation,

$$F = f.m.$$

If the force continue to act for a time t, the velocity produced will be ft, and by multiplying each side of the above equation by t, we have

$$\begin{aligned} Ft &= (ft)m \\ &= vm, \end{aligned}$$

or the product of the force and the time during which it acts is equal to the product of the velocity generated and the mass.

Now in many cases the velocity generated by a force is known when neither the magnitude of the force nor its time of action can be determined.

It is necessary, therefore, to give a name to the quantity of motion produced in any time, known or unknown, and it is termed momentum.

The momentum of a moving body is the product of its mass and velocity.

The relation between the force F, the time t during which it acts and the momentum vm produced in the time t is given by the equation

$$Ft = vm.$$

Hence, a force is measured by the momentum it gene-

rates in a unit of time, and the whole amount of force expended in any time is equal to the momentum generated in that time.

49. Impulsive Forces.—When the force generates a finite velocity in a time so short that its duration cannot be perceived by the senses, it is termed an impulsive force, and is measured by the whole momentum produced by it.

When motion is communicated to a billiard ball or a cricket ball by a blow, and when a shot is projected from a gun, the forces are of this kind.

The action in such cases is not instantaneous. When a ball is fired from a gun it is evident that the expansive force of the gunpowder acts on the ball during the time it moves along the bore, and when bodies are moved by impact, experiment shows that they are in contact for a finite interval of time, during which they tend to penetrate each other, their forms being first changed and then restored.

The time of action of an impulsive force is too short to be measured, although it is of finite duration; hence, an impulsive force cannot be measured in the same way as a finite force, namely, by the momentum which would be generated in a second if the force continued to act for the second; its magnitude must be estimated by the whole momentum produced, irrespective of the time taken to produce it. Two impulsive forces which produce equal momenta are considered equal, although they might prove unequal if they could be measured in the same way as finite forces; for the times of action, though both inappreciably small, may not be the same. This method of estimating impulsive forces never leads to inconvenience, for all the impulsive forces with which we are acquainted in Nature produce finite momenta, and we are never called upon to consider the intermediate conditions. Nor have we to deal

with ordinary and impulsive forces at the same time, for during the brief time of action of the impulse, the effect produced by an ordinary force is inappreciable compared with that of the impulse. It must, however, be remembered that a difference exists, and a rule established, for finite forces cannot be applied to impulsive forces unless the effect of this difference has been considered. There are, however, relations between the force and change of momentum true for all forces, and some of these we proceed to consider.

50. Proposition V.—*In the uniformly accelerated motion of a material particle, the product of the force and the time of action is equal to the change of momentum in the time.*

Let a force F act on a mass m for a time t, and let u and v be respectively the initial and final velocities, then

$$\text{F}=fm,$$

$$\text{and } v=u\pm ft;$$

therefore by eliminating f we have

$$\text{F}t=\pm(vm-um).$$

51. Deductions.—If the force acts in the direction of motion, the positive sign is to be taken, and then

$$\text{F}t=vm-um=\text{the } gain \text{ of momentum.}$$

If the force acts in the direction opposite to that of motion, so as to be of the nature of a resistance, then the negative sign is to be taken, and

$$\text{F}t=um-vm=\text{the } loss \text{ of momentum.}$$

From the first of these equations we see that if a body starts from rest so that $u=0$, then

$$\text{F}t=vm=\text{the momentum produced.}$$

And from the second, if a body is brought to rest by a resistance F in the time t so that $v=0$, then

$$\text{F}t=um=\text{the momentum destroyed.}$$

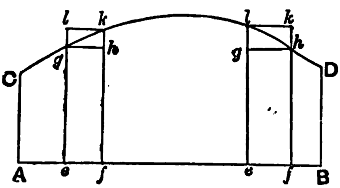
Hence, the momentum generated by a force F in time t may be destroyed by the same force in time t. To find how long a body having a given momentum will move against a given resistance, we have therefore the equation

$$\text{time} = \frac{\text{momentum}}{\text{resistance}}.$$

In the above equations we have supposed the force to be constant during the time t. If the force be variable we may suppose the time to be divided into very small portions, through each of which the force is constant, or we may proceed as in the following proposition :—

52. Proposition VI.—*A force* F *acts for a time* t, *in the same direction, but with varying intensity, to represent graphically the momentum generated.*

Let AB represent the time t, and let CD be a curve such that the ordinate eg at any point g represents the force F at the instant corresponding to e, then will the area ABDC represent the momentum generated in the time t.



Suppose the time AB to be divided into small intervals, and imagine two other forces M and N to act for the same time, remaining constant throughout each interval and receiving a sudden increase or decrease at the end of each interval. Let the first force, M, be equal in magnitude to F at the beginning of each interval when F is increasing, and at the end of each interval when F is decreasing, and let N be equal to F at the end of each interval when F is increasing, and at the beginning when F is decreasing. Then F will always be greater than M and less than N, except at the instants in which all are equal; hence, however small the intervals of time may be, the area

representing the momentum generated by F must be greater than that representing the momentum of M and less than that representing the momentum of N, that is, the required area must lie between the sum of the rectangles like *eh*, and the sum of the rectangles like *ek*.

Hence the momentum generated by the force F in the time t must be represented by the area ABDC.

Example 23.—A body weighing 8 pounds has a velocity of 483 feet per second, and is retarded by a force equal to the weight of half a pound; how long will it move?

Let the measure of the mass be 8, then the measure of the resistance is $\frac{1}{2}g$, and equation $Ft = um$ becomes

$$\tfrac{1}{2}gt = 483 \times 8.$$

$$\text{If } g = 32{\cdot}2, \therefore t = 240 \text{ seconds.}$$

Example 24.—A train weighing 50 tons moves up an incline rising 1 in 100, the resistance being 10 pounds per ton; in what time will the speed decrease from 30 to 15 miles per hour?

The mass of the train is 50×2240, or 112,000 pounds.

The pressure exerted down a plane rising 1 in 100 by a body weighing 112,000 pounds is equal to the weight of 112,000 pounds $\div 100$.

The absolute measure of this force is $1120g$.

The absolute measure of the resistance is $500g$.

The effective force is therefore $(500 + 1120g)$, or $1620g$.

The velocity lost is 15 miles per hour, or 22 feet per second.

Hence the momentum lost is $112{,}000 \times 22$ per second.

Consequently if t be the time,

$$1620gt = 112{,}000 \times 22;$$

$$\therefore t = 47{\cdot}5 \text{ seconds.}$$

Example 25.—A mass W moves on a horizontal plane under the action of a force P resisted by another force R; find the velocity acquired when W has moved from rest for t seconds.

Let W, P, and R be expressed in pounds.

If the measure of the mass be W, the absolute measures of the weight and the forces will be respectively Wg, Pg, Rg.

Momentum exerted by $P = Pgt$;

Momentum spent in overcoming force of R pounds for t seconds $= Rgt$;

Momentum accumulated $= Wv$.

$$\therefore Pgt = Rgt + Wv;$$

$$\therefore v = gt\frac{P-R}{W}.$$

Example 26.—Q draws up P by a cord passing over a pulley without weight; find the velocity when Q has descended for t seconds.

To vary the exercise we will express the masses and the weights in gravitation measure.

The momentum exerted $= Qt$.

The momentum spent in overcoming force of P pounds for t seconds $= Pt$.

The momentum accumulated $= \frac{Pv}{g} + \frac{Qv}{g}$;

$$\therefore Qt = Pt + \frac{Pv}{g} + \frac{Qv}{g};$$

$$\therefore v = gt\frac{Q-P}{Q+P}.$$

Example 27.—To one end of a flexible string passing over a fixed pulley a weight of 8 pounds is attached, and to the other end two weights of 4 pounds and 6 pounds respectively; if the 4 pounds be removed after the 10 pounds has been descending for 6 seconds; find how much longer the 9 pounds will descend before coming to rest.

First.—Take 1 pound of mass as the unit of mass; then there are g units of force in the weight of 1 pound, and in the example the force on one side is $10g$ and on the other $8g$. Now the momentum generated by a force of $10g$ in 6 seconds $= 60g$.

The momentum expended in overcoming a force of $8g$ for 6 seconds $= 48g$.

Hence, the momentum accumulated is $12g$, but this is equal to mv, where m is the whole mass moved, or 18 pounds, and v the velocity;

$$\therefore 18v = 12g, \text{ or } v = \frac{2}{3}g.$$

Second.—Let the weight move t seconds after the removal of the 4 pounds.

The momentum destroyed is $8gt-6gt$, but this is equal to the momentum accumulated, or to $\frac{2}{3}g \times 14$;

$$\therefore 8gt-6gt=\frac{2}{3}\cdot 14g$$

$$t=\frac{14}{3}=4\frac{2}{3} \text{ seconds.}$$

SECTION II.—*Energy.*

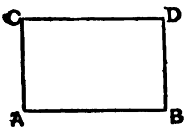
53. For many purposes the elements of the effect of a force which have to be considered are the magnitude of the force and the *space* through which its point of application moves. For instance, if a ball made to impinge on a mass of clay penetrates it to a certain depth, there must have been accumulated in the ball at the instant of striking the mass a power reprèsented by the resistance offered by the tenacity of the clay and the distance through which this resistance is overcome. We have shown that there are cases in which we know the product of the force and time, or the change of momentum produced, without having either factor; there are also cases in which we require to deal with the product of a force and a space without determining the factors.

We require, therefore, other terms which will enable us to consider force in its relation to the space, irrespective of the time through which it acts. The terms work and energy supply what is required.

54. Definition of Work.—A force does work when its point of application moves in the direction of the force; and the work done is measured by the product of the magnitude of the force and the distance through which the point of application moves in the direction of the force.

For example, if a body be allowed to fall from any height, the work done by the earth's attraction on the body will be found by multiplying its weight by the height. If a load be lifted through a height s, the work done by the lifting force is the weight of the load multiplied by s. If a locomotive pulling with a uniform force F, move a train through a distance s, the work done by the locomotive is Fs.

Hence, work may be represented graphically by an area. Let AB represent the line of action of the force acting on the point A, and let AB be the space moved through. Draw AC perpendicular to AB, to represent the magnitude of the force; then the work done= F.s=AC.AB=area ABDC.



55. Definition of Energy.—The power of a machine or moving body to do work is termed Energy.

To distinguish the energy of motion from a kind of energy to be subsequently described, the former may be called *kinetic* energy. Hence, the kinetic energy of a moving body is its power to do work in consequence of its actual velocity.

To say, therefore, that the energy of a system of bodies is 100, is to signify that the same mechanical effects can be obtained from the system as can be obtained by the fall of a weight of 100 from a unit of height, or, regarding the weight as constant, by the fall of a unit-weight from a height of 100 units.

56. Proposition VII.—*In the uniformly accelerated motion of a material particle, the work of the force in a given time is equal to the variation of the kinetic energy in this time.*

Let m be the mass of the body, F the force, f the accele-

ration, s the space through which the body moves, u, v the initial and final velocities respectively, then

$$F=fm,$$

$$\text{and } v^2=u^2\pm 2fs.$$

Eliminating f between the two equations, we have

$$Fs=\pm\left(\frac{v^2m}{2}-\frac{u^2m}{2}\right).$$

57. Deductions.—If the force acts in the direction of motion the positive sign must be taken, and since $\frac{1}{2}u^2m$ is the kinetic energy of the body at the beginning of the space s, and $\frac{1}{2}v^2m$ that at the end, we have

$$Fs=\text{gain of kinetic energy.}$$

If the force acts in the direction opposed to that of motion, the negative sign must be taken, and

$$Fs=\tfrac{1}{2}u^2m-\tfrac{1}{2}v^2m=\text{the loss of kinetic energy.}$$

If the body starts from rest so that $u=0$, then $Fs=$the kinetic energy produced. If the body starts with velocity u, and is retarded by a force F, then, s being the space in which it is brought to rest, that is to say, at the end of which $v=0$, we have

$$\begin{aligned} Fs&=\tfrac{1}{2}u^2m \\ &=\text{the kinetic energy accumulated at the beginning of the motion.}\end{aligned}$$

Hence the kinetic energy generated by a force F acting through a space s may be destroyed by the same force in the space s. To find how far a body having a given kinetic energy will move against a given resistance, we have, therefore, the equation—

$$\text{the distance}=\frac{\text{kinetic energy}}{\text{resistance}}.$$

In the above equations we have supposed the force to be constant through the space s. If the force be variable we may divide the space s into portions s' so short, that we

may suppose the force constant through each, and to receive a slight sudden increase at the end; then for each small space s',

$$Fs' = \text{gain or loss of energy in space } s'.$$

By adding the different portions of the work on the one side, and the different increments of energy on the other, we arrive at the result that the total work done by the force is equal to the total increase of kinetic energy. Or, we may proceed to prove this more exactly by the following proposition.

58. **Proposition VIII.**—*A force* F *acts always in the same direction, but with varying intensity, to represent graphically the work done, while the point of application of the force moves in a straight line through a distance* s.

Let AB represent the space s (see figure of Proposition VI.), and let CD be a curve such that the ordinate eg at any point g represents the force F at the point e, then will the area ABDC represent the work done.

Suppose the space AB divided into small parts, and imagine two other forces M and N to act through the same space, remaining constant throughout each small part, and receiving a sudden increase or decrease at the end of each. Let the first force M be equal to F at the beginning or end of each small space, and let the second N be equal to F at the end or beginning of each small space according as F is increasing or decreasing. Then F will always be greater than M and less than N, except at the points at which they are equal; hence, however small the divisions may be, the area representing the work done by F must be greater than the area representing the work done by M, and less than that representing the work done by N, that is, the required area lies between the sum of the rectangles like eh and the sum of the rectangles like ek.

Hence, the work done by F in the space s must be represented by the area ABDC.

59. Arbitrary Units of Work.—From the foregoing definition of work it follows that the unit of work must be the work done by the unit of force while its point of application moves through a unit of length.

The unit of work usually employed by English engineers is a gravitation unit termed the *foot-pound*. It is the work required to overcome the attraction of the earth on 1 pound of matter through 1 foot, in other words, to lift 1 pound a foot high. It varies with the weight of the pound, and is therefore different in different localities, but, like the gravitation unit of force, is very convenient where absolute accuracy is not required. Measured in these units, the work required to lift 5 pounds 12 feet high is 60 foot-pounds; to lift a ton 2 yards high, 2240×6, or 13,440 foot-pounds.

The French gravitation unit of work is the *kilogrammetre*, or the work spent in raising a kilogramme through a height of 1 metre. As a kilogramme is 2·2055 pounds, and a metre is 3·2809 feet, a kilogrammetre is $2{\cdot}2055 \times 3{\cdot}2809$, or 7·233 foot-pounds.

In engineering, a larger unit of work is required; and as not only *quantity of work*, but also *rate of working* has to be considered, the engineering unit is coupled with a unit of time. The performance of 33,000 foot-pounds in one minute is termed a *horse-power*.

60. Absolute Unit of Work.—The absolute unit of work is by the definition of work, the work done by the absolute unit of force acting through a unit of space; hence since there are 32·2 British absolute units of force in the weight of a pound where g is 32·2, there are 32·2 British absolute units of work in the foot-pound. Similarly there are 9·8 French absolute units of work in the kilogrammetre at Paris.

Example 28.—To one end of a flexible string passing over a fixed pulley a weight of 8 pounds is attached, and to the other end two weights of 4 pounds and 6 pounds respectively; if the 4 pounds be detached after the 10 pounds has been descending through 12 feet, find how much further the 6 pounds will descend before coming to rest.

Let a pound be the unit of mass, then there will be g units of force in the weight of 1 pound. Hence, the force on one side is $10g$, and on the other $8g$. Now the work done by a force of $10g$ acting through 12 feet is $120g$.

The work done in overcoming a force of $8g$ through 12 feet is $96g$.

Hence the accumulated work is $24g$.

But this is equal to $\frac{1}{2}mv^2$ when m is the whole mass moved and v the velocity;

$$\therefore v^2 = 24g \div 9 = 2\tfrac{2}{3}g.$$

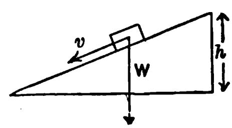
Now let the weights move through s feet after the removal of the 4 pounds.

The work done is $8gs - 6gs$, or $2gs$.

But this is equal to the energy accumulated in the 14 pounds of mass.

$$\therefore \tfrac{1}{2}.14v^2 = 2gs,$$
$$\therefore 7 \times 2\tfrac{2}{3}g = 2gs,$$
$$\therefore s = 9\tfrac{1}{3} \text{ feet.}$$

Example 29. A weight W descends an inclined plane whose height is h; required the velocity with which it reaches the bottom.



Let W be the weight in pounds, then $W \div g$ is the gravitation measure of the mass.

The energy exerted $= Wh$ and the whole is stored.

$$\therefore Wh = \frac{Wv^2}{2g};$$
$$\therefore v^2 = 2gh.$$

This is evidently independent of the inclination of the plane, and is the same when the surface down which the body falls is curved instead of plane.

Example 30.—A weight W moves on a horizontal plane under a force P, resisted by another force R; find the velocity acquired when W has moved from rest through a space s.

Take the forces and masses in gravitation measure.

$$\text{Energy exerted by } P = Ps.$$

$$\text{Work done} = Rs.$$

$$\text{Work accumulated} = \frac{Wv^2}{2g}.$$

$$\therefore Ps = Rs + \frac{Wv^2}{2g},$$

$$v^2 = \frac{2gs(P-R)}{W}.$$

Example 31.—Q draws up P by a cord passing over a pulley without mass; find the velocity when Q has descended s feet.

Here Q descends and P ascends through s feet.

Let Q and P be the weights in pounds of the bodies, and let the masses and weights be expressed in absolute measure; then their masses are P and Q and their weights gP and gQ.

$$\text{Energy exerted} = gQs.$$

$$\text{Work done} = gPs.$$

$$\text{Work accumulated} = \tfrac{1}{2}Pv^2 + \tfrac{1}{2}Qv^2.$$

$$\therefore gQs = gPs + \tfrac{1}{2}Pv^2 + \tfrac{1}{2}Qv^2.$$

$$\therefore v^2 = 2gs\frac{Q-P}{Q+P}.$$

Example 32.—A horse pulls a tram-car on a horizontal road with a constant force equal to one-tenth of the weight of the car; if the horse be unhooked when he has moved over 100 yards, the car will move 60 yards more before it will stop. What will the speed of the car be in miles per hour if the horse continue to pull with the same force for 180 yards, supposing the resistance to remain the same?

Let the weight be W pounds and resistance R pounds.

The force $\left(\frac{W}{10}\right)$ acting through 300 feet exerts energy which will overcome the resistance R for 160 yards or 480 feet.

$$\text{Hence, } 30W = 480R.$$

Let v = the velocity in feet per second when the force acts through 540 feet.

$$\text{Then energy exerted} = 54\text{W}.$$

$$\text{Work done} = 540\text{R} = 33\tfrac{3}{4}\text{W}.$$

$$\text{Work accumulated} = \frac{\text{W}v^2}{2g}.$$

$$\therefore 54\text{W} = 33\tfrac{3}{4}\text{W} + \frac{\text{W}v^2}{64}.$$

$$\therefore \quad v^2 = 64 \times 20\tfrac{1}{4},$$

$$v = 8 \times 4\tfrac{1}{2} \text{ ft. per sec.}$$

$$= 24\tfrac{6}{11} \text{ miles per hour.}$$

Example 33.—Find the horse-power of an engine which is to move at the rate of 21 miles per hour up an incline which rises 1 in 120, the weight of the engine and load being 48 tons, and the resistance 10 pounds per ton.

21 miles an hour is 1848 feet per minute.

The resistance is 480 pounds. Hence, the work done in overcoming resistance is 1848×480 foot-pounds per minute.

The load is raised per minute $1848 \times \frac{1}{120}$ or 15·4 feet. Hence, the work done in lifting the load is $15{\cdot}4 \times 48 \times 2240$ foot-pounds per minute.

$$\text{Therefore, the horse-power} = \frac{1848 \times 480 + 154 \times 10752}{33000},$$

$$= 77{\cdot}056.$$

EXERCISE VI.

1. Define the momentum of a moving body; and show what effects are measured by means of it.

2. Define the energy of a moving body. How is it measured? Should the energy or the momentum be used as a measure of the *force* of a moving body?

3. A ball, weighing 8 pounds, is rolled on a grass-plot with a velocity of 12 feet a second; if the resistance of the ground be $\frac{1}{10}$th of the pressure on it, how far will the ball roll?

4. What are the usual arbitrary units of energy, and what is the British absolute unit?

5. How long would a body weighing 20 pounds, and having an initial velocity of 20 feet per second, move against a resistance of 1 pound?

6. How is it that a ball of lead can be thrown further than a ball of cork of the same size?

7. A shot, weighing 30 pounds, is fired from a gun weighing 3 tons, and leaves the gun with a velocity of 1500 feet per second; find the velocity of the gun's recoil.

8. A body weighing 20 tons moves with a uniform speed of 20 miles an hour on a level road, the resistances to the motion averaging 12 pounds per ton; how much work is done per hour?

9. How is the energy of a moving body estimated? Through what distance must a force equal to the weight of $\frac{1}{2}$ a pound act on a mass of 48·3 pounds in order to increase the velocity from 24 feet to 36 feet per second?

10. To one end of a string hanging over a pulley a mass of 5 ounces is attached, and to the other end two masses of 3 ounces and 4 ounces respectively. The masses are allowed to move for 3 seconds, and then the 4 ounce is removed. Find how long and how far the 5 ounce will continue to ascend.

11. Find the tension of a rope which draws a carriage of 8 tons weight up a smooth incline of 1 in 5, and causes an increase of velocity of 3 feet per second.

12. If on the same incline the rope breaks when the carriage has a velocity of 48·3 feet per second; how far will the carriage continue to move up the incline?

13. The weights at the extremities of a string which passes over the pulley of an Atwood's machine are 500 and 502 grammes. The larger weight is allowed to descend; and 3 seconds after motion has begun, 3 grammes are removed from the descending weight. What time will elapse before the weights are again at rest?

14. If a ball weighing 4 pounds be thrown on a horizontal plane with a velocity of 100 feet per second, and the friction between the ball and plane be $\frac{1}{8}$ the weight; find the distance to which the ball will go.

15. A weight W of 12 pounds on a rough table is attached to a thread which passes over the edge of the table, and sustains a weight of 3 pounds; when the latter has descended through 5 feet the

thread breaks, and W moves through 4 feet more and comes to rest; what is the friction?

16. A certain engine, if not attached to a train, could get up a speed of 60 miles an hour in 2 minutes; how long would it take to get up a speed of 40 miles an hour when attached to a train of twice its own weight, supposing the acceleration due to gravity 32 feet per second, the friction to be $\frac{1}{100}$ of the weight, and all resistances except friction to be neglected?

17. In the above example, if the steam be shut off when the speed is 40 miles an hour, how long and how far will the train move before it stops; and how far would the engine move without the train?

18. The last carriage of a railway train gets loose whilst the train is running at the rate of 30 miles an hour up an incline of 1 in 150. Supposing the effect of friction upon the motion of the carriage to be equivalent to a uniformly retarding force equal to $\frac{1}{300}$ of the weight of the carriage; find (i.) the length of time during which the carriage will continue running up the incline; and (ii.) the velocity with which it will be running down after the lapse of twice this interval from the instant of its getting loose.

19. Two bodies, weighing 1 and 2 pounds respectively, are connected by an inelastic string 18 feet in length, which passes over a smooth pulley; the two bodies are lifted up to the pulley, so that the string is slack; if they are let fall simultaneously, find the time that will elapse before the 1-pound weight returns to the pulley.

20. A locomotive, with train attached, runs steadily down an incline of 1 in 200 at a rate of 60 miles an hour; it would run up the same incline at a rate of 40 miles per hour. On the supposition that the frictional forces which oppose the motion of the train vary as the velocity, find the gradient up which the train would run at 10 miles an hour.

21. Find the horse-power of a locomotive which moves a train weighing 50 tons at the rate of 30 miles per hour on a road where the resistances amount to 1-70th of the weight.

22. Find the horse-power of an engine which is to move at the rate of 30 miles an hour up an incline rising 1 in 100, the weight of engine and load being 44 tons, and resistance 15 pounds per ton.

23. Find at what rate an engine of 45 horse-power could draw a train weighing 60 tons up an incline rising 1 in 224, the resistance amounting to 11 pounds per ton.

24. A weight of 6 ounces is drawn up 4 feet along the lid of a smooth desk by a weight of 5 ounces, which, attached to the other weight by a string, hangs over the top of the lid and descends vertically. The lid is inclined at such an angle that the rise is 2 in 9. Find the velocity acquired when the heavier weight reaches the top of the lid.

25. A train moving at the rate of 23 miles an hour is brought to rest in 3 minutes, the retarding forces being supposed uniform during the time. If the whole weight of the train be $94\frac{1}{2}$ tons, show that the resultant of the retarding forces is $\cdot55$ tons ($g = 32\cdot2$).

26. A locomotive weighing 10 tons setting out from rest acquires a velocity of 20 miles an hour after running through a mile on a horizontal plane under the action of a constant pressure P, and retarded by a constant resistance R. Find, in pounds, the difference $P - R$.

CHAPTER VI.

THE LAWS OF MOTION.

61. Introduction.—In order to apply strict mathematical reasoning to dynamical phenomena, it is necessary to state as axioms the results of the experiments described in Chapter III., together with other fundamental propositions which may be illustrated by experiment, but the truth of which must be admitted without demonstration. These dynamical axioms were first simply and concisely stated by Newton in forms which we shall refer to as Newton's Laws of Motion.

62. The First Law of Motion.—*A body preserves its state of rest or of uniform motion in a straight line unless compelled to change its state by external forces.*

This law may be paraphrased entirely or in part as follows :—

(i.) There are no causes residing in a body which can influence its motion, and the state as regards motion of a body when all external forces cease to act, is quite independent of its size or nature.

(ii.) Matter has no inherent property by which when in motion it would naturally come to rest, or by which its state of rest or motion is changed; but it has a property termed its *inertia*, that when not acted on by any external forces it continues in the same state, either of rest or of

uniform rectilinear motion. The first law of motion affirms the inertia of matter, and is therefore sometimes called the Law of Inertia.

The law states that a body whose motion is not affected by any forces, moves through equal spaces in equal times; hence *equal times are those in which a body not acted on by any force passes through equal spaces.* In Chapter I., § 4, we took the earth in its rotation about its axis as the body whose uniform motion serves to measure time; that is to say, we assumed that there are no forces tending to alter the period of the earth's rotation about its axis.

63. The Second Law of Motion.—*Change of motion is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.*

The facts implied by negation in the second law are as important as those actually affirmed, and as they are more likely to escape observation we will mention them first.

(i.) *The law applies equally to rest and motion.*—No distinction is made between bodies already moving, but in all cases change of motion is proportional to the impressed force, the change being in the direction of the force. Hence, when a force or forces act on a body in motion, the change of motion in magnitude and direction is the same as if the forces acted on the body at rest. If, therefore, a body already in motion be acted on by a force, a velocity will be produced in the direction of the force, which must be compounded with the initial velocity to give the resultant velocity.

(ii.) *The law applies to each of a number of forces.*—The statement applies not only to one, but to every force acting on a body; hence the effects produced by any one or more forces acting on a body are independent of any other forces that may be in action. Whether other forces are acting or not, any force produces a change of motion proportional to its magnitude and having its direction.

(iii.) *Relation between Force, Acceleration, and Mass.*—The change of motion produced is measured by the product of the mass and change of velocity (§ 40), so that as long as the mass and the interval of time considered remain the same, the force is proportional to the change of velocity; hence, constant force produces constant acceleration in its own direction. The law states that change of motion is *proportional* to the impressed force, and by taking the change of motion in the unit of time and properly selecting our units, we may write that the impressed force is *equal* to the momentum generated in a unit of time. This is the fact expressed by the equation on page 53, $F = fm$.

(iv.) *Definition of Equal Masses and Equal Forces.*—The second law affords definitions of equal masses and equal forces.

Equal masses are those in which the same force produces equal velocities in equal intervals of time.

Equal forces are those which give to equal masses equal velocities in equal intervals of time.

(v.) *Superposition of Equilibrium.*—If a number of forces which are made to act on a body originally at rest produce no motion, then they would not produce change of motion if they were applied to a body in motion. Such forces are said to be in equilibrium; hence, forces in equilibrium may be applied to or taken from any body at rest or in motion, whether acted on by other forces or not, without altering its state as regards motion.

(vi.) *Transmissibility of Force.*—The effect of a force depends only on its magnitude and line of action, and not on its point of application; hence, any point in the body which is in the line of action of the force may be supposed to be the point of application.

(vii.) *Composition of Forces.*—Forces acting on the same body produce accelerations proportional to themselves and

having their directions; hence, if a number of forces act simultaneously on a body, the resultant force is proportional to the resultant acceleration, and has the same direction; hence, forces may be compounded or resolved exactly like accelerations. If two forces when acting on a body produce no change of motion, the velocity which one would generate in any interval of time must evidently be equal and opposite to that which the other would generate in the same time; and hence, by the above definition, the forces are equal.

When any number of forces act on a body and produce no change of motion, any one must evidently be equal and opposite to the resultant of all the others; hence, if we know that force which would neutralise the effect of a number of given forces so that when it is added to the system there is no change of motion by simply reversing it, we obtain at once the resultant of the given forces.

Using the term "impressed force" for the resultant of the applied forces, we have the following statements:—

The first law states that where there is no impressed force there is no change of motion.

The second law states that where there is an impressed force there is change of motion.

The contrapositive statements, necessarily true if these are true, are—

When there is change of motion there must be impressed force;

And when there is no change of motion there is no impressed force.

64. The Third Law of Motion.—*To every action there is always an equal and opposite reaction, or, the mutual actions of two bodies are always equal and in opposite directions.*

Newton gave the following illustrations of the third law:—

(1.) If any one presses a stone with his finger, his finger is pressed with an equal force in the opposite direction.

This is an illustration of the statement that when one surface rests against another, the action of the first on the second is equal and opposite to that of the second on the first. The surfaces have a common normal at the point of contact, and the actions are along the common normal.

(2.) A horse drawing a body by means of a rope is pulled backwards by a force equal to that with which the body is drawn forwards.

In general terms, when one body acts on another by means of a rope or cord, the actions are equal and opposite.

Every point of the cord must therefore be acted on by equal and opposite forces which are equal to the actions at the extremities.

In other words, the force transmitted directly by a string or rod without weight is transmitted without change, or the tension of a string or rod is the same throughout.

(3.) When one body changes the motion of another by impact, its own motion is changed by the same amount and in the opposite direction, for at each instant during the impact the forces exerted between them are equal and opposite.

Change of motion here means the same as in Chapter IV.; that is to say, change of momentum, *i.e.* mass multiplied by change of velocity.

(4.) When two bodies, A and B, attract one another, the force with which A attracts B is equal and opposite to the force with which B attracts A.

EXERCISE VII.

1. A spring-balance bearing a weight is suspended in the car of a balloon, and when at rest the index points to 100 on the graduated scale; what will be the indication—

(a.) When the balloon is ascending with uniform velocity?

(b.) When it ascends with velocity, increasing in a second by 6·44 feet per second?

(c.) When it descends uniformly?

(d.) When it descends with velocity which diminishes at the rate of 5·36 feet per second in a second?

(e.) When it descends with velocity which increases at the rate of 8·05 feet per second in a second ($g=32·2$)?

2. A weight of 10 pounds is suspended by a string from the roof of a railway carriage which moves uniformly; what is the direction and the tension of the string?

If the train rush over a precipice, what will be the direction and tension of the string during the fall?

3. If a heavy particle be dropped down a well directed towards the centre of the earth, which side of the well will the particle strike in consequence of the earth's rotation?

4. An iron cage starts from rest to descend the shaft of a mine. The tension produced by the cage in the supporting chain is 200 pounds, whereas if the cage were at rest it would be 225 pounds. Find the time of descending 100 feet.

5. A weight of 400 pounds moving on a smooth horizontal surface has a string attached to it, which passes horizontally over a pulley, and then vertically to a weight of 10 pounds, which rests on a surface below. If a velocity of 12 feet per second be given to the larger weight, how high will the smaller weight be lifted by it?

6. In the above example what must the smaller weight be that the larger may be brought to rest in a space of 6 feet.

7. A balloon ascends with a uniformly accelerated velocity so that a weight of 1 pound produces on the hand of the aeronaut sustaining it a downward pressure equal to that which 17 ounces would produce at the earth's surface; find the height which the

balloon will have attained in one minute from the time of starting, not taking into account the variation of the accelerating effect of the earth's attraction.

8. I jump off a platform with a 20-pound weight in my hand. What will be the pressure of the weight upon my arm while I am in the air? Give full reasons for your reply.

9. Show how the Third Law of Motion may be used to find the tension of the string and the acceleration when one ball is drawn up an inclined plane by another which hangs by a string passing over a fixed pulley at the top of the plane.

10. Show how the Third Law of Motion holds in the case of a stone which is in the act of falling towards the earth.

CHAPTER VII.

PARALLELOGRAM OF FORCES.

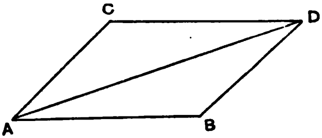
65. Introduction.—By combining the parallelogram of accelerations and the Second Law of Motion, we can immediately deduce the important proposition known as the *parallelogram of forces.* When forces act on the same particle they are proportional to the accelerations they would severally produce on the particle; consequently there are propositions on the composition of forces corresponding to all those on the composition of accelerations given in Section 18.

By assuming portions of the laws of motion, the parallelogram of forces may be proved independently of the parallelogram of velocities, but the demonstration is more cumbrous and less general than that which depends on the latter proposition. Duchayla's proof, which is given in the Appendix to this chapter, for the reference of any reader who may be interested in it, is one of the simplest answering this description.

66. Proposition XII.—*If two forces acting on a particle be represented in magnitude and direction by two adjacent sides of a parallelogram, the resultant of these forces will be represented*

in magnitude and direction by that diagonal of the parallelogram which passes through the particle.

Let AB and AC represent in magnitude and direction two forces acting on a mass at A, and let AD be the diagonal of the parallelogram of which AC and AB are the sides.



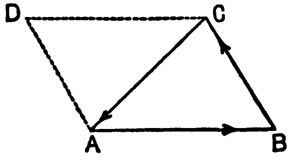
Proof.—Since the forces act on the same mass, the velocities which they generate in a unit of time will be proportional to the forces, and will, therefore, also be represented by the lines AB and AC respectively. But by the parallelogram of velocities, the velocities represented by AB and AC are equivalent to a single velocity represented by AD. Therefore the two forces produce on the mass at A in a unit of time a velocity represented by AD, and are therefore equivalent to a force represented by AD.

67. Deductions from the Parallelogram of Forces.

(i.) If the component forces are at right angles, the resultant is the square root of the sum of their squares.

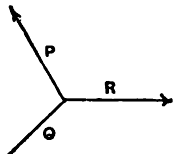
(ii.) If two forces are represented by AB, BC, the resultant is represented by the third side AC of the triangle ABC.

Proof.—A force represented by BC is also represented by AD, which is parallel and equal to BC, and the resultant of forces represented by AD and AB is represented by AC.



(iii.) When three forces acting on a particle can be represented in magnitude and direction by the three sides of a triangle *taken in order*, they are in equilibrium.

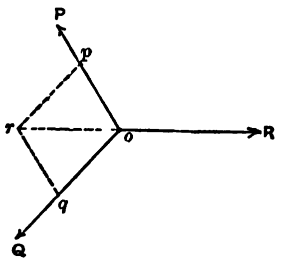
Let ABC be the triangle; let PQR be three forces proportional to the sides BC, CA, AB, acting on a particle, P parallel to BC, Q parallel to CA, and R parallel to AB; then the forces will be in equilibrium.



Proof.—For draw AD parallel to BC, and CD parallel to BA. Forces represented by AB and AD in magnitude and direction will have a resultant represented by AC in magnitude and direction. Therefore forces represented by AB, AD, and CA in magnitude and direction will be in equilibrium. AD is equal and parallel to BC; hence the forces represented by AB, BC, and CA are in equilibrium.

(iv.) The converse of this proposition is also true. When three forces acting on a particle are in equilibrium, the sides of any triangle which are parallel to the lines of action of the forces are also proportional to the forces.

Proof.—Let forces P, Q, R acting on a particle at O keep at rest. In the direction of P take any point p, and in the direction of Q take a point q, such that $Oq : Op :: Q : P$.



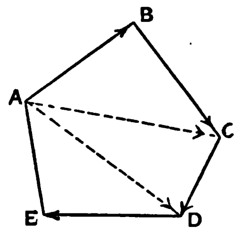
Complete the parallelogram $Oprq$. Then Or represents the resultant of P and Q in magnitude and direction, and must, therefore, have the direction of R. Hence, any triangle having its sides parallel to the directions of the forces being similar to the triangle Opr, will therefore have its sides proportional to the forces.

Again, it is a theorem in geometry that if there be two triangles, such that the sides of one are respectively perpendicular to those of the other, then these sides are propor-

tional; hence, in the above proposition, if the lines be drawn *perpendicular* to the directions of the forces they will be proportional to the forces.

(v.) The resultant of forces represented by the sides of a closed polygon ABCDE taken in order is zero.

Proof.—For forces represented by AB, BC are equivalent to a force represented by AC; forces AC, CD to a force AD; and the resultant of forces represented by AD, DE, EA is zero by (iii.)



(vi.) The resultant of forces represented by all the sides but one of a closed polygon taken in order is represented by the remaining side taken in the direction opposed to that of the components.

Proof.—For if a force be added equal to the remaining side taken in the same order as the other components, there will be no resultant; hence, if this force be reversed, it must be equivalent to all the others.

(vii.) Any given force may be replaced by two components in any two directions, provided the components can be represented in magnitude and direction by the adjacent sides of a parallelogram, the diagonal of which represents the given force.

If the components X, Y be parallel respectively to two lines at right angles, and if the force F makes angles α and $90°-\alpha$ with these lines, then by the parallelogram of forces,

$$F^2 = X^2 + Y^2,$$

$$X = F\cos\alpha;\ Y = F\sin\alpha.$$

Hence we see that the component of a force in a direction making an angle α with that of the force (the other component being at right angles to the first) is found by multiplying the force by the cosine of α.

(viii.) If any number of forces, F_1, F_2, F_3, etc., act on a particle in directions making angles α_1, α_2, α_3, etc., with a fixed straight line Ox, their resultant can be found by first resolving each parallel and perpendicular to the fixed straight line; secondly, adding the components in each direction; and, lastly, making the two sums respectively equal to X and Y, the components of the resultant in the two directions.

Proof.—For

F_1 may be replaced by $F_1 \cos\alpha_1$ and $F_2 \sin\alpha_1$;

F_1 ,, ,, $F_2 \cos\alpha_2$ and $F_2 \sin\alpha_2$; and so on.

Substituting these components for the forces, we see that the whole system is equivalent to

$$F_1 \cos\alpha_1 + \text{etc.} \ldots + F_n \cos\alpha_n \text{ parallel to } Ox;$$

$$\text{and } F_2 \sin\alpha_2 + \text{etc.} \ldots + F_n \sin\alpha_n \text{ perpendicular to } Ox.$$

But if X, Y be the components of the resultant R in the two directions, the whole system, which is by the definition of resultant equivalent to R, is also equivalent to

$$X \text{ parallel to } Ox,$$
$$Y \text{ perpendicular to } Ox.$$

$$\text{Hence, } X = \Sigma F \cos\alpha,$$
$$Y = \Sigma F \sin\alpha,$$
$$\text{and } R^2 = X^2 + Y^2,$$
$$= (\Sigma F \cos\alpha)^2 + (\Sigma F \sin\alpha)^2.$$

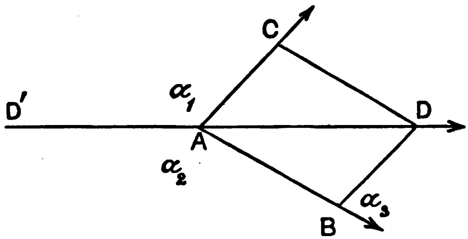
If θ be the angle made by the direction of the resultant with Ox,

$$\tan\theta = \frac{Y}{X} = \frac{\Sigma F \sin\alpha}{\Sigma F \cos\alpha}.$$

In order that there may be equilibrium, that is to say, that R may be zero, we must evidently have $\Sigma F \sin\alpha = 0$, and $\Sigma F \cos\alpha = 0$.

(ix.) If two forces be given in magnitude and direction, the resultant can be found by the solution of a triangle.

Let AB, AC represent forces F_1, F_2, and let the angle CAB be α_3. Complete the parallelogram ABDC, and let the resultant AD be R. Let F_3 be a force represented by AD′ equal and opposite to R, then F_1, F_2, F_3, are three forces in equilibrium.



Let $\angle$ CAD′ $=\alpha_1$ and $\angle$ BAD′ $=\alpha_2$. Consider the $\triangle$ ADB; it has its sides respectively proportional to R, F_1, and F_2. Now, by a well-known trigonometrical formula (or Euclid II. 12 and 13),

$$\begin{aligned} AD^2 &= AB^2 + BD^2 - 2AB.BD.\cos ABD; \\ &= AB^2 + BD^2 + 2AB.BD.\cos\alpha_3; \\ \therefore R^2 &= F_1^2 + F_2^2 + F_1.F_2 \cos\alpha_3. \end{aligned}$$

Also in any triangle ADB the sides are proportional to the sines of the opposite angles; that is,

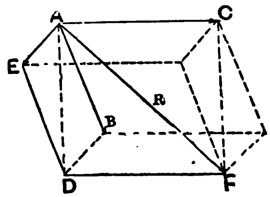
$$AD : AB : BD :: \sin ABD : \sin ABD : \sin BAD.$$

$$\begin{aligned} \text{But } \sin ABD &= \sin(\pi - \alpha_3) = \sin\alpha_3, \\ \sin ADB &= \sin(\pi - \alpha_1) = \sin\alpha_1, \\ \sin BAD &= \sin(\pi - \alpha_2) = \sin\alpha_2. \end{aligned}$$

$$\text{Hence } \frac{F_3}{\sin\alpha_3} = \frac{F_2}{\sin\alpha_2} = \frac{F_1}{\sin\alpha_1}.$$

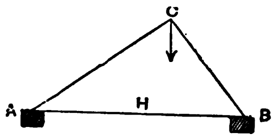
Hence, when three forces acting on the same point are in equilibrium, if each be divided by the sine of the angle between the other two, the three quotients will be equal.

68. (x.) **Parallelopiped of Forces.**—If three forces, F_1, F_2, F_3, acting on a point be represented in magnitude and direction by the three sides EA, BA, CA of a parallelopiped, their resultant will be represented in magnitude and direction by the diagonal of the parallelopiped through the point of application.



Proof.—The resultant of F_1 and F_2 is represented by the diagonal of the parallelogram EB, and the resultant of this force and F_3 is represented by AF, the diagonal of the parallelogram ADFC.

69. Remarks and Examples on the Preceding Propositions.—(i.) It may not be unnecessary to remark that the parallelogram, triangle, or polygon, representing the forces, are not the bodies acted on, but are auxiliary figures drawn on paper, so that their sides represent forces *acting on the same point.* But sometimes when the figure representing the body and its material connections is drawn, a triangle in the figure happens to have its sides parallel to the forces, and may therefore be used to furnish the numerical relation of the forces. The triangle and polygon of forces afford exceedingly useful methods of representing graphically the stresses in the various parts of frames and structures. Forces acting along a rod tending to compress it are termed *thrusts;* forces tending to stretch it are termed *tensions;* both are termed *stresses.*

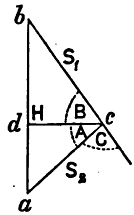


Example 34.—ACB is a frame supported at A and B and loaded at C by a weight W; find the stresses S_2, S_1, and H on AC, BC, and AB, and the pressures on A and B.

Draw a vertical line *ab* to represent W; from *a* and *b* draw lines

respectively parallel to AC and BC and meeting in c, and through c draw cd perpendicular to ab.

Then the sides of $\triangle\, abc$ being respectively parallel to the directions of the forces acting at C, are proportional to these forces, and for similar reasons the sides of $\triangle\, acd$ are proportional to the forces at A, and the sides of $\triangle\, bdc$ to the forces at B.



$$\text{Hence } \frac{H}{S_2}=\frac{dc}{ac}=\cos A,$$

$$\frac{H}{S_1}=\frac{dc}{bc}=\cos B,$$

$$\frac{W}{S_1}=\frac{ab}{bc}=\frac{\sin C}{\cos A};$$

$$\therefore \frac{S_1}{\cos A}=\frac{S_2}{\cos B}=\frac{W}{\sin C}.$$

Again, if P_2 and P_1 be the pressures on the supports at A and B,

$$\frac{P_1}{S_1}=\frac{bd}{bc}=\sin B,$$

$$\text{or } P_1=\frac{W\cos A.\sin B}{\sin C};$$

$$\frac{P_2}{S_2}=\frac{ad}{ac}=\sin A,$$

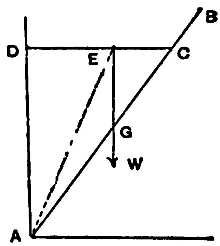
$$\text{or } P_2=\frac{W\sin A.\cos B}{\sin C}.$$

(ii.) *Three forces on a rigid body.*—Whenever three forces which are not parallel act on a body and keep it at rest, their directions pass through the same point, and, consequently, the triangle of forces can be applied to them.

Proof.—Since the forces were in equilibrium, the resultant of any two must be equal and directly opposite to the third; but this resultant lies in the same plane as its two components, and passes through their point of intersection. Hence, the third force also lies in the same plane and passes through the same point.

This fact alone is usually sufficient to determine the position of equilibrium of the body.

Example 35.—A uniform beam AB, 12 feet long and weighing 40 pounds, rests with one end A at the bottom of a vertical wall and a point C, 2 feet from the other end, connected by a horizontal string CD to a point D, 8 feet above A; find the reaction at A, and the tension of the string.



The beam is supported by three forces—

(i.) The tension of the string T along CD.

(ii.) The weight W vertically through G, 6 feet from A, its direction meeting CD in E.

(iii.) The reaction R at A along AE.

Now $\triangle$ AED has its sides respectively parallel to the three forces, and therefore proportional to them.

$$\text{Since } AD=8 \text{ and } AC=10, \therefore DC=6 \text{ feet};$$
$$\text{and } DE : DC :: AG : AC,$$
$$\text{or } DE : 6 :: 6 : 10;$$
$$\therefore DE=3{\cdot}6.$$

Knowing DE and AD, we can find $AE=8{\cdot}77$.

From the triangle of forces, $AD : DE :: W : T$,
$$\text{or } 8 : 3{\cdot}6 :: 40 : T;$$
$$\therefore T=18 \text{ lbs.}$$
$$\text{Also } AD : AE :: W : R,$$
$$\text{or } 8 : 8{\cdot}77 :: 40 : R;$$
$$\therefore R=43{\cdot}8 \text{ lbs.}$$

Example 36.—Two strings, AB, CB, 8 and 15 inches long respectively, are attached to points A and C in the same horizontal line and support a weight of 51 ounces at B; the angle formed by the strings is a right angle. Find the tensions of the strings.

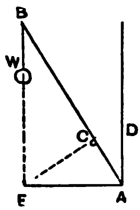
Here the sides of $\triangle$ ABC are respectively perpendicular to the three forces acting at B, consequently they are proportional to the forces.

$$AC^2=8^2+15^2=17^2.$$

Let P be the tension in AB, and Q the tension in CB.
$$\text{Then } AC : BC :: W : P,$$
$$\text{or } 17 : 15 :: 51 : P;$$
$$\therefore P=45 \text{ ounces.}$$
$$\text{Also } AC : AB :: W : Q,$$
$$\text{or } 17 : 8 :: 51 : Q;$$
$$\therefore Q=24 \text{ ounces.}$$

Example 37.—A rod 8 feet long, the weight of which has not to be considered, is placed across a smooth horizontal rail, and rests with one end against a smooth vertical wall, the distance of which from the rail is 1 foot; the other end of the rod bears a weight of 12 pounds. Find the position of equilibrium and the pressure on the rail.

The three forces acting on the rod AB, are the weight W, at B vertical, the reaction of the wall at A horizontal, and the pressure on the rail C perpendicular to AB. Since the directions of the first two meet in E, the third must be along EC. Hence the position of equilibrium is such that EC is perpendicular to AB. Draw CD horizontal.



$$\text{Now } \frac{CD}{CA} = \frac{CA}{AE} = \frac{AE}{AB}.$$

Since each is the cosine of $\angle$ EAB.

$$\therefore CA^2 = AE.CD = AE \ (\because CD = 1),$$

$$\text{and } AE^2 = CA.AB = 8CA;$$

$$\therefore CA = 2 \text{ and } AE = 4.$$

$$\text{Hence } \cos \angle EAB \text{ is } \tfrac{1}{2}; \text{ and } \therefore \angle EAB = 60^\circ.$$

Again $\triangle$ EAB has its sides respectively perpendicular to the directions of the three forces acting on the rod, AE being perpendicular to the direction W, and AB perpendicular to the direction of the pressure P on the rail at C, and BE perpendicular to the pressure at A.

$$\text{Hence } AE : AB :: W : P,$$

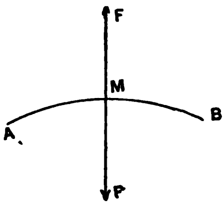
$$\text{or } 4 : 8 :: 12 : P;$$

$$\therefore P = 24 \text{ pounds.}$$

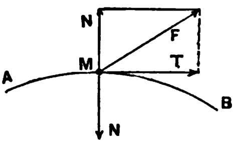
(iii.) *Constrained Bodies.*—When motion can occur only in a given direction, other motion being prevented by the resistance called into action perpendicular to this direction, the only other condition of equilibrium necessary is that the sum of the resolved parts of the forces in the given direction shall be zero.

The body is then said to be constrained. Suppose, for example, a horse pulls a railway car by means of a chain which makes an angle a with the direction of the rails, the force tending to move the car is the force exerted by the

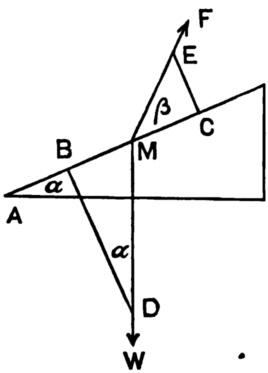
horse multiplied by cosine α. Again, if a ring M be supported on a metallic rod bent into the form of an arc AB, and a force F be applied at the point M perpendicularly to the curve, the ring will not be moved, for the force makes equal angles with the two directions in which the point might be displaced, and there is no reason to suppose that motion could occur in one direction more than in the other direction. The reaction of the curve must therefore be equal to F.



Suppose, now, a force F applied at M in another direction; this force may be resolved into two, one T along the tangent to the curve, and the other N along the normal (*i.e.* the perpendicular to the tangent). The component T tends to displace the point, while the component N is counteracted by a reaction equal and opposite to it.



(iv.) *Inclined Plane.*—A body M, of weight W, is supported on a smooth inclined plane by a force F making an angle β with the plane; to find the ratio of F to W, and the ratio of the reaction R to W.



Let the angle of the plane be α. Along the direction of F take ME to represent F in magnitude, and draw EC perpendicular to the plane; and along the vertical through M take MD to represent W, and draw DB perpendicular to the plane; then MC represents the force tending to move M up the plane, and MB the

force tending to move M down the plane. Hence, for equilibrium,

$$MC = MB, \text{ or } F \cos\beta = W \sin\alpha.$$

Again, EC represents the force tending to lift M away from the plane, and BD the force pressing M against the plane. Hence, the difference BD—EC represents the reaction of the plane; in other words,

$$R = W \cos\alpha - F \sin\beta.$$

If the plane be *rough*, the reaction is found in the same way, and we still have

$$R = W \cos\alpha - F \sin\beta;$$

but we no longer necessarily have MB equal to MC, but

$$MB = MC \pm \text{a line representing the friction.}$$

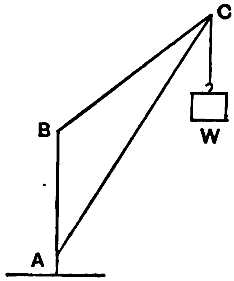
If the greatest value of the friction be a certain fraction (μ) of the pressure on the plane, then MB must lie between

$$MC \pm \text{a line representing } \mu R,$$

or $W \sin\alpha$ lies between $F \cos\beta \pm \mu R$.

EXERCISE VIII.

1. A weight W of 2 tons is suspended from the extremity C of a crane ABC; the crane-post from A to B is 20 feet long, the tension-rod BC is 30 feet, and the jib AC 40 feet; find the stresses along BC and AC.



2. If two lines, AB, CA, represent two forces acting on a point, the one towards the point and the other from it, show how to find the resultant.

3. Three pegs, A, B, and C, are driven in a wall at the corners of a right-angled isosceles triangle, of which the base AC is horizontal. A cord passes over the three pegs and supports two weights of 20 pounds attached to its ends; find the pressures on the pegs.

4. Three forces acting on a point are represented by lines of 12,

24, and 15 inches, including angles of 60° ; find the length of the line which will represent the resultant.

5. The angles between three forces of 42, 52, and 10 pounds respectively are 120° ; find the resultant.

6. Show that if three forces acting on a point be represented in magnitude and direction by the three lines drawn from the middle points of the sides of a triangle to the opposite angles, the forces are in equilibrium.

7. Along the sides of an equilateral triangle ABC three forces, each equal to the weight of 1 pound, act in direction as follows :— From A to B, from A to C, and from B to C ; find their resultant.

8. A particle placed in the centre of a square is acted on by forces of 1, 2, 3, and 4 pounds respectively, tending to the angular points ; find the magnitude and direction of the resultant force.

9. Replace two forces of 20·3, and 39·6 kilogrammes, respectively, acting at right angles by two others also acting at right angles, the larger being 40 kilogrammes.

10. Three rods meet at a point and form a tripod to sustain a weight ; show how to find the ratios of the pressures on the rods.

11. Three forces, 25, 60, and 72 grammes, having directions at right angles to one another, act upon a point ; find their resultant.

12. A cord is attached to two fixed points A and B in the same horizontal line, and bears a ring weighing 10 pounds at C, so that ACB is a right angle ; find the tension in the cord.

13. Two cords, $AC=44$ inches, $BC=117$ inches, are attached to points A and B in the same horizontal line and to a weight of 10 pounds at C. The angle ACB is a right angle ; find the pressures on A and B.

14. Prove that if two forces be represented by two diagonals of a parallelogram, their resultant will be represented by a line equal to twice one of the sides of the parallelogram.

15. Find the resultant of six forces, 1, $\sqrt{2}$, 2, 3, and $\sqrt{3}$ pounds, acting on a point, the angles taken in order being 45°, 75°, 60°, and 90°.

16. A cord CB has one end C attached to a point in a vertical wall, and the other end B to the extremity of a beam AB which rests against the wall. If $AC=5$ feet 5 inches, $AB=8$ feet 1 inch,

and weighs 130 pounds; find the length of CB, neglecting the friction between the wall and beam.

17. A weight is supported by two strings which are attached to it, and to two points in a horizontal line; if the strings are of unequal length, show that the tension of the shorter string is greater than that of the other.

18. Three forces, represented by those diagonals of three adjacent faces of a cube which meet, act at a point; show that the resultant is represented by twice the diagonal of the cube.

19. A picture-frame is hung over a smooth peg; show how to compare the tension of the string with the weight of the frame, and find how the tension is affected by increasing the length of the string.

20. A uniform heavy rod, weighing 50 pounds and measuring 9 feet in length, is placed across a smooth rail and rests with one end against a smooth wall, the distance between the rail and wall being 16 inches; find the position of equilibrium and the pressure on the rail.

21. Two small rings slide on the arc of a smooth vertical circle; a string passes through both rings and has three equal weights attached to it, one at each end and one between the rings; find the position of the rings when they are in equilibrium.

22. If on an inclined plane the pressure, force, and weight be as the numbers 28, 17, and 25; find the inclination of the plane to the horizon, and of the force to the plane.

23. A and B are two given points in a horizontal line 1 foot apart; to A a string AC is fastened $=\frac{1}{2}$AB; to B another string is fastened, which, passing through a ring at C, supports a weight W at its other extremity; show that $\angle B = 90° - 2\angle A$.

24. A weight W is supported on a smooth inclined plane by three forces, each equal to $\frac{1}{3}$ W, which act one vertically upwards, another horizontally, and the third parallel to the plane; show that if θ be the inclination of the plane, $\tan\frac{\theta}{2} = 2$.

25. Four rods are jointed together so as to form a trapezoid ABCD; and the frame thus formed is placed in a vertical plane, with the base AB resting on a horizontal plane, and the side DC parallel to AB. If a load of 12 cwt. is placed at each of the upper

corners D and C, find the stresses produced along the rods, the lengths being as follows :—

AB = 22 feet, DC = 12 feet, AD = BC = 13 feet.

Point out which of the stresses are thrusts, and which are tensions.

APPENDIX TO CHAPTER VI.

Duchayla's Proof of the Parallelogram of Forces.—*Axioms.* (i.) A force may be translated to any point in its line of action without altering its effect.

(ii.) A system of equilibriating forces may be super-imposed on a body without producing any effect.

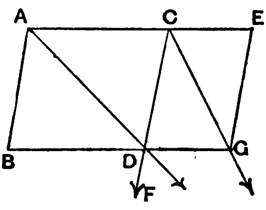
(iii.) Two forces acting on a particle, if not in equilibrium, will have a single resultant through the particle.

Proposition 1.—To find the *direction* of the resultant of two forces acting upon a point.

(i.) When the forces are equal, it is clear that the direction of the resultant will bisect the angle between the directions of the forces ; or, if we represent the forces in magnitude and direction by two lines drawn from the point where they act, the diagonal of the parallelogram described on these lines will be the direction of the resultant.

(ii.) Let us assume that the proposition is true for two unequal forces p and m, and also for p and n. We can prove that it must then necessarily be true for two forces p and $m+n$.

Let AB, AC be the directions, and be proportional to the forces p and m in magnitude ; complete the parallelogram BC, and draw the diagonal AD ; then, by hypothesis, the resultant of p and m acts along AD.



Again, take CE in the same ratio to AC that n bears to m. By axiom (i.) we may suppose the force n to act at A or C, and therefore the forces p, m, and n in the lines AB, AC, and CE are the same as p and $m+n$ in the lines AB and AE.

Now, replace p and m by their resultant, and transfer its point of application from A to D; then resolve this force at D into two, parallel to AB and AC; these resolved parts must evidently be p and m, p acting in the direction DF, and m in the direction DG. Transfer these two forces, p to C and m to G.

But by the hypothesis, p and n acting at C have a resultant in the direction CG; let, then, p and n be replaced by their resultant, and transfer its point of application to G.

But m acts at G.

Hence, by this process we have, without disturbing the equilibrium, removed the forces p and $m+n$ which acted at A to the point G.

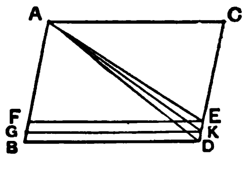
Therefore the resultant of p and $m+n$ must pass through G, and acts in the direction of the diagonal AG, provided our hypothesis is correct.

But the hypothesis is correct for equal forces, as p, p; from the above it is therefore true for forces p, $2p$; consequently, for p, $3p$, and so it is true for p, rp.

Hence, it is true for p, rp and p, rp and, consequently, for $2p$, rp, and so forth; and it is finally true for sp and rp, where r and s are positive integers.

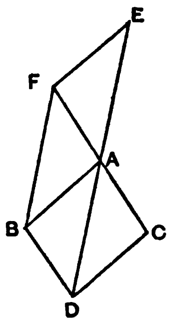
(iii.) We have still to show that the Proposition is true for incommensurable forces.

Let AB, AC represent two such forces. Complete the parallelogram BC. Then if their resultant do not act along AD, suppose it to act along AE; draw EF parallel to BD. Divide AC into a number of equal portions, each less than DE; mark off from CD portions equal to these, and let K be the last division; K evidently falls between D and E: draw GK parallel to AC. Then the two commensurable forces represented by AC, AG have a resultant in the direction AK, from the preceding proof; and this is nearer to AG than the resultant of the forces represented by AC, AB, which is absurd, since AB is greater than AG.



In the same manner we may show that every direction except AD leads to an absurdity, and therefore the resultant must

act along AD, whether the forces be commensurable or incommensurable.



2. To find the *magnitude* of the resultant.

Let AB, AC be the directions of the given forces, AD that of their resultant; take AE opposite to AD, and of such a length as to represent the magnitude of the resultant. Then the forces represented by AB, AC, AE balance each other. Complete the parallelogram BE.

Then AC must be in the same straight line with AF, the resultant of AB and AE; hence, FD is a parallelogram; and therefore AE = FB = AD.

Or the resultant is represented in *magnitude* as well as in *direction* by the diagonal of the parallelogram.

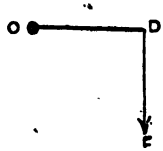
CHAPTER VII.

MOMENTS OF FORCES.

SECTION I.—*General Principles.*

68. Introduction.—The moment of a force about a point is its tendency to produce rotation about the point.

Suppose a rod OD, capable of turning about the fixed point O, to be acted on by a force F; experiment shows that the tendency of the force to turn the rod about O depends on the magnitude of the force and on its distance from the point O. We might, for example, double this tendency, either by doubling the force, or by keeping the force the same and causing it to act at twice the distance from O. Hence the tendency of the force to turn the rod about O is measured by the product of the force into the perpendicular OD.



When one point in a body is fixed in order that the body may be at rest, it is evident that the moment in one direction about that point must be equal to the moment in the opposite direction.

But when a body is at rest under the action of forces it is evident that we may imagine any point in it to be fixed, for the fixing of a point which is already at rest without introducing any stress at the point would not disturb the equilibrium. Hence, the tendency to turn in one direction

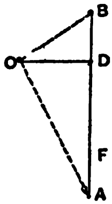
about *any* point must be equal to that in the opposite direction; in other words, the algebraical sum of the moments of the forces about *any* point must be zero. This principle, which is an exceedingly useful one, admits of rigid deduction from the parallelogram of forces, and its demonstration and application form the subject of the present chapter.

69. Definition and Graphic Representation of the Moment of a Force —*The moment of a force with respect to a point or a line is the tendency of the force to move the body on which it acts about the point or the line.*

The product of any force F *by the perpendicular from a point on to the line of its direction is the measure of the moment of the force with respect to the point.*

If a line be drawn through a point perpendicular to the plane of the point and the force, the same product measures the moment of the force about the line.

Thus, if AB represent the force F, the moment of F about O is AB$\times$OD. This product is numerically *twice the area of the triangle having the line representing the force for base and the given point for apex.*



We show, therefore, that the moments of two forces are equal when we prove the equality of the triangles formed by joining the extremities of the lines representing the forces to the given point.

It is evident that the moment of a force about a point in its own direction is zero, for then the area is zero.

It is convenient to consider moments in one direction, as, for example, that of the hands of a clock as positive, and moments in the opposite direction as negative.

The moment of a force is therefore a product, the sign of which depends on the signs of the factors. Thus in Figs. I.

and IV. the moments Fa about the point A are positive, and in Figs. II and III. negative.

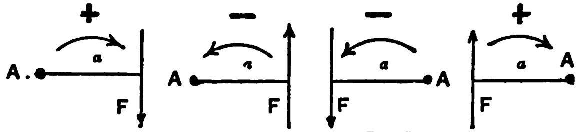
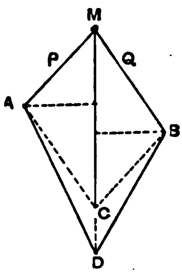


FIG. I. FIG. II. FIG. III. FIG. IV.

70. **Proposition X.**—*When two forces act on a particle, the moments about a point in the direction of the resultant are equal.*

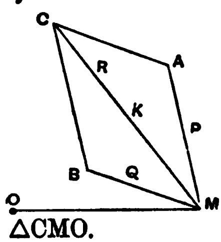
Let M be the particle and P and Q the forces. Complete the parallelogram, and take any point D in the resultant, we shall prove that the moments of P and Q about this point are equal by proving that the triangles AMD, BMD are equal. Now the perpendiculars from A and B on MC are equal, and we may regard MD as the common base; hence, the triangles having the same base and equal heights are equal, and therefore the moments of the forces about the point D are equal.



The above is a particular case of the following proposition:—

71. **Proposition XI.**—*The moment of the resultant of two forces acting on a particle about a point in their plane is equal to the algebraical sum of the moments of the forces.*

Let MB and MA represent the forces, and MC their resultant.



Let O be the point.

The moment of BM about O is twice ΔBMO.

The moment of AM about O is twice ΔAMO.

The moment of CM about O is twice ΔCMO.

Now these triangles have the same base MO, consequently they are proportional to their heights.

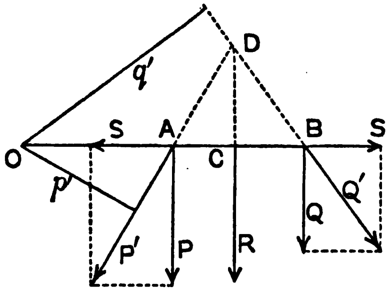
Let the heights of A, B, and C be respectively a, b, and c.

Let K be the point of intersection of the two diagonals of the parallelogram; then since K bisects each diagonal, its distance from OM is the arithmetical mean of the distances of the extremities of either diagonal. Hence,

$$\text{Distance of } \mathrm{K} = \tfrac{1}{2}c,$$
$$\text{also distance of } \mathrm{K} = \tfrac{1}{2}(a+b).$$
$$\therefore c = a+b$$
$$\therefore \Delta\mathrm{CMO} = \Delta\mathrm{BMO} + \Delta\mathrm{AMO}.$$
$$\therefore \text{moment of } \mathrm{R} = \text{moment of } \mathrm{P} + \text{moment of } \mathrm{Q}.$$

72. Proposition XII.—*The algebraical sum of the moments of two parallel forces acting on a rigid body about any point in their plane is equal to the moment of their resultant.*

Let P and Q be parallel forces acting on a rigid body, and R their resultant. Let O be any point, and let the perpendiculars from O to the directions of P and Q meet them in A and B respectively, and the direction of R in C.



Suppose two equal and opposite forces S, S to be applied at the points A and B along the line AB, and compounded with P and Q giving respectively the resultants P′ and Q′, the directions of which meet in a point D.

The addition of the forces S, S will evidently neither affect the resultant, nor the moments about O; *i.e.* R is also the resultant of P′ and Q′.

Let p' and q' be respectively the arms of P′, Q′ with regard to O; then, by Proposition X.,

$$\mathrm{P}'.p' + \mathrm{Q}'.q' = \text{moment of } \mathrm{R}.$$

But $P'.p' = P.AO$, for these products are respectively twice the areas of triangles on the same base OA, and having the same height.

$$\text{Similarly } Q'.q' = Q.BO;$$
$$\therefore P.AO + Q.BO = \text{moment of } R.$$

73. Deductions from Proposition XII.—If the point O coincide with C, then

$$P.AC - Q.BC = 0,$$
$$\text{or } \frac{AC}{BC} = \frac{Q}{P},$$

that is to say, any point C in the direction of the resultant divides the distance AB between P and Q in a constant ratio; consequently R must be parallel to P and Q, and OC is therefore perpendicular to it, and the moment of R about O is R.OC.

$$\therefore P.AO + Q.BO = R.CO.$$

If the point O coincide with A, we have

$$R.AC = Q.AB,$$
$$\therefore \frac{R}{AB} = \frac{Q}{AC} = \frac{P}{BC};$$

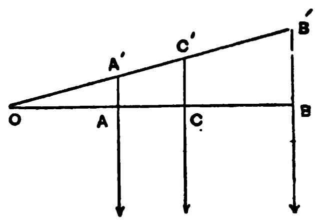
therefore, by the principles of proportion,

$$\frac{R}{AB} = \frac{P+Q}{AC+BC}.$$

Since the denominators are equal,

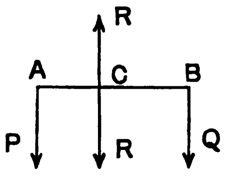
$$\therefore R = P + Q.$$

If OA′C′B′ be another line from O meeting the directions of P, Q, and R in A′, B′, C′, then the segments of this line are proportional to the segments of the line AB. Hence,



$$\frac{R}{A'B'} = \frac{P}{B'C'} = \frac{Q}{A'C'}.$$

If a force equal and opposite to R be applied to the body, there will be equilibrium, and any one of the forces P, Q, and R will be equal and opposite to the resultant of the other two.



From the two equations,

$$R=P+Q,$$

$$\frac{R}{AB}=\frac{P}{CB}=\frac{Q}{AC}$$

(the latter of which is easily remembered on account of its symmetry), we see that *if two parallel forces act on a body, their resultant is their sum when they act in the same direction* (like P and Q), *and their difference when they act in opposite directions* (like P and R or Q and R), *and if the forces and their resultant be each divided by the distance between the other two, the three quotients will be equal to one another.*

74. Proposition XIII.—*To find the magnitude and position of the resultant of any number of parallel forces in the same plane.*

Let F_1, F_2, . . . F_n be the forces, those acting towards one direction being positive, and those in the opposite direction being negative. Let a line AB be drawn so as to intersect the directions of all the forces. Take a point of reference E on AB, and let d_1, d_2, . . . d_n be the distances of the forces from the point E measured along AB, distances to the right being positive, and those to the left of E negative. Let R_2 be the resultant of the first two forces, R_3 the resultant of the first three, and so on, and let r_2, r_3 be the corresponding distances from E.

By § 64,

$$F_1+F_2=R_2,$$

$$R_2+F_3=R_3,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$R_{n-1}+F_n=R_n.$$

By adding these equations and remarking that R_2, R_3, . . . R_{n-1} will occur on both sides of the result, and will therefore cancel, we obtain

$$F_1+F_2+F_3 \; . \; . \; . \; +F_n=R_n. \qquad (1.)$$

Again, by § 73 we also have

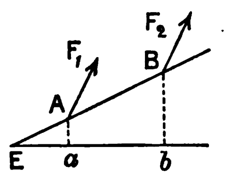
$$F_1.d_1+F_2.d_2=R_2.r_2,$$
$$R_2.r_2+F_3.d_3=R_3.r_3,$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$R_{n-1}.r_{n-1}+F_n.d_n=R_n.r_n.$$

By adding these equations, we obtain

$$F_1.d_1+F_2.d_2 \; . \; . \; . \; +F_n.d_n+=R_n.r_n. \qquad (2.)$$

75. Proposition XIV.—*To find the magnitude and position of the resultant of a number of parallel forces which act on a rigid body.*

Let F_1, F_2, . . . F_n, be the forces; it may be shown precisely as above that $R=\Sigma \cdot F$. Let the distances of the the points of application from a fixed plane be p_1, p_2, . . . p_n. Let F_1, F_2 act at points A and B. Let a plane through AB, perpendicular to the plane of reference, cut that plane in ab, and let the lines AB, ab, meet in E; then, if their resultant R_2 act at C, by § 73, F_1.AE+F_2.BE=R_2.CE; but if r_2, r_3, . . . r_n are the distances of the points of application of the successive resultants from the plane of reference,



$$BE : AE : CE :: p_2 : p_1 : r_2.$$

Therefore $\quad F_1p_1+F_2p_2=R_2r_2.$

Similarly $\quad R_2r_2+F_3p_3=R_3r_3,$

$$. \quad . \quad . \quad . \quad . \quad . \quad .$$

$$R_{n-1}r_{n-1}+F_np_n=R_nr_n.$$

By adding, therefore, we obtain

$$F_1 p_1 + F_2 p_2 + \ldots + F_n p_n = R_n r_n.$$

$$\therefore \Sigma F.p = r.R.$$

This equation determines r, the distance of the point of application of the resultant from the plane of reference.

If x, y, z be the values of p for three planes at right angles, and $\bar{x}$, $\bar{y}$, $\bar{z}$ the corresponding values of r,

$$\Sigma Fx = \bar{x}.R;\ \Sigma Fy. = \bar{y}.R;\ \Sigma Fz = \bar{z}.R.$$

These equations determine $\bar{x}$, $\bar{y}$, and $\bar{z}$, and therefore the position of the point.

76. Centre of Parallel Forces.—It is important to notice that the equations are not altered by changing the directions of the forces if the points of application are kept in the same relative positions; hence, *the resultant of a system of parallel forces acting at different points in a rigid body passes through a fixed point, the position of which is independent of the direction of the forces.* This point is termed the centre of the forces.

77. Hints for Solution of Problems.—In applying the principles of this chapter to the solution of problems, the following summary of facts and hints may be found useful:—

(i.) When two forces are in equilibrium, they must act in the same straight line.

(ii.) When three forces not parallel are in equilibrium, their lines of direction must meet in a point.

(iii.) Reactions of surfaces are perpendicular to the surfaces.

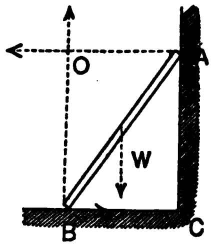
(iv.) When there are two unknown forces, an equation may be found containing only one, either by taking moments about some point in the other, or by resolving the forces in a direction at right angles to this other.

(v.) When there are three unknown forces, one which is not required may be excluded, for two equations may be found by resolving in a direction perpendicular to that of

this unknown force, and by taking moments about a point in the line of its direction.

(vi.) Generally, in forming equations for the solution of problems, resolve at right angles to unknown reactions, and take moments about points in directions common to as many as possible.

Example 38.—A uniform beam AB, 17 feet long, and weighing 120 pounds, rests with one end against a smooth wall and the other end on a smooth floor, this end being tied by a string 8 feet long to a peg at the bottom of the wall; find the tension of the string.



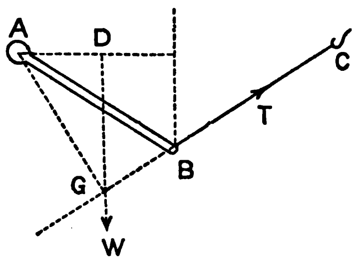
Four forces act on the beam—a horizontal reaction at A, a vertical reaction at B, the weight of the beam, which we may suppose to act vertically through its middle point, and the tension of the string.

Now, the directions of the two reactions intersect in a point O; consequently, by taking moments about O, an equation will be obtained which will not contain the reactions. We then have 120 pounds at an arm equal to $\frac{1}{2}$ BC or 4 feet, and the tension T at an arm equal to BO or AC, which is $\sqrt{(17^2-8^2)}$ or 15 feet.

$$\text{Hence, } 15T = 120 \times 4.$$

$$\therefore T = 32 \text{ pounds.}$$

Example 39.—A beam A has one end attached to a hinge A, and the other end attached to a cord BC. The weight of the beam is 50 pounds, and may be supposed to act at its middle point. The beam and cord make angles of 60° on opposite sides of the vertical line through B. Find the tension of the cord.



Here there are two unknown forces, the reaction R of the peg, and the tension of the cord T. If, however, we take moments about the

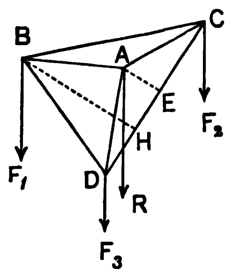
hinge, we shall obtain an equation not involving R, for the moment of R about this point is zero.

Let $2l$ = the length of the beam; draw a vertical line through the middle point of AB to represent the direction of the weight. The perpendicular on this line is $\frac{\sqrt{3}}{2}\cdot l$; hence the moment of the weight about A is $25\sqrt{3}.l$.

The perpendicular on the direction of the cord is $l\sqrt{3}$, and therefore the moment of the tension is $T.l\sqrt{3}$.

$$\text{Therefore } T.l = 25.l,$$
$$\text{and } T = 25.$$

Example 40.—Required to decompose a force F applied at A into three parallel forces F_1 F_2 F_3 applied at three given points, B, C, D, in a plane, which is not parallel to the direction of the force F.



Join the points A, B, C, and D, and draw BH and AE perpendicular to DC.

Take moments about the line DC.

$\therefore F_1 \cdot BH = F \cdot AE.$

But $BH : AE = \triangle BCD : \triangle ADC$.

$$\therefore \frac{F_1}{\triangle ADC} = \frac{F}{\triangle BCD}.$$

By taking moments about BD and BC, similar equations are obtained for F_2 and F_3.

$$\therefore \frac{F}{\triangle BCD} = \frac{F_1}{\triangle ACD} = \frac{F_2}{\triangle ABD} = \frac{F_3}{\triangle ABC}.$$

If the force F be represented by the area of the triangle BCD, the components will be represented by the areas ACD, ABD, and ABC.

EXERCISE IX.

1. A uniform beam, 3·4 feet long and weighing 4 pounds 13 ounces, rests with one end against a smooth wall and the other on a smooth table, and is prevented from sliding by a string which, passing over a smooth pulley at the edge of the table, bears a weight of 18 ounces. Find the distance of the end of the beam from the wall.

2. A rod, AB, attached to a hinge at A has the end B tied to a cord passing over a pulley placed vertically over the hinge, and supporting a weight equal to half that of the rod. The distance between the hinge and pulley and the length of the rod are each 6 feet. Find the inclinations of the rod and the cord, and the direction of the pressure on the hinge.

3. Two smooth small pegs are in the same horizontal line 3 feet apart, and a uniform rod 12 feet long is placed over one and under the other. Find the position of the rod when the pressure on one peg is four times that on the other.

4. If the corners of a square whose side is 2 feet are acted on by parallel forces of 5, 7, 9, and 11 pounds respectively; find the distances of the point of application of the resultant from the sides of the square.

5. A uniform rod, 18 inches long and weighing 18 ounces, is laid symmetrically across two pegs in the same horizontal line 10 inches apart. Where must a weight of 5 ounces be placed that the pressure on one of the pegs may be 8 ounces?

6. A rigid body immersed in water is acted on by parallel forces of 8, 4, 2, and 6 pounds applied respectively at points in the body whose depths are 3, 7, 9, and 5 feet. Find the depth of the point of application of their resultant.

7. A cord AC, CB is attached to a peg C, and has its ends A and B tied to the ends of a rod AB 130 inches long. At A and B weights of 7 pounds and 3 pounds respectively are suspended, and the points A and B then rest at distances of 77 and 127 inches respectively below a horizontal line through C; find the lengths and tensions of the strings, neglecting the weight of the rod.

8. A cord CB has one end C attached to a point in a vertical wall, and the other end B to the extremity of a beam AB, which rests against the wall. If AC=5 feet 5 inches, AB=8 feet 1 inch, and weighs 130 pounds, find the length of CB, neglecting the friction between the wall and beam.

9. A circular disc, of radius 65 inches, in a vertical plane is moveable about an axis through the centre. From the extremities of two radii at right angles to each other, weights of 24 and 10 pounds respectively are suspended; find the depths of the points of suspension below the horizontal line through the centre.

10. A rectangular table is supported in a horizontal position by four legs at its four angles: a given weight W being placed upon a given point of it, show that the pressure on each leg is indeterminate; and find the greatest and least value it can have for a given position of the weight.

11. A thread 12 feet long is fastened at points A and B in the same horizontal line 8 feet apart. At C and D, points 4 feet and 5 feet respectively from A and B, weights are attached; what must be the ratio of the weights that CD may be horizontal?

12. Three weights, of 4, 5, and 6 pounds respectively, are suspended over the circumference of a circular hoop, by three strings knotted together at its centre; determine the relative positions of the strings when the hoop supported at its centre remains horizontal.

13. If a straight rod be balanced on a point, and weights of 1, 2, and 3 pounds be suspended at distances of 6, 12, and 18 inches from the point in one direction, and 2, 3, 4 pounds at distances of 4, 10, and 12 inches in the other; find where a weight of 1 pound must be placed so as to keep the rod at rest.

14. Three rods, 7, 6, and 9 feet long respectively, are jointed together, and when laid across two pegs A and B, rest in one horizontal straight line; find the distance AB and the strains at the joints.

15. A beam AB rests with one end A against a smooth vertical wall, and the other end B on a smooth horizontal plane; it is prevented from sliding by a cord tied to one end of the beam and to a peg at the bottom of the wall; the length of the beam is 10·6 feet, and the length of the string 9 feet. Suppose the weight of the beam to be 112 pounds and to act vertically through its middle point; find the forces acting on the beam.

16. A beam AB is placed with one end A inside a hemispherical bowl, and with a point C in it, resting on the edge of the bowl; find the inclination of the beam, friction being neglected.

17. If the radius of the bowl be 10 feet, and the beam make an angle of 30° with the horizon, find the length of the beam.

18. ABC is a triangle capable of moving about the right angle B in a vertical plane; find the ratio of the weights which must be attached to A and C, that the side AC may be horizontal, having given that AB = 6·15 and AC = 9·53 inches.

19. A uniform beam AB rests with the upper end B on a prop, and the lower end A attached to a string, which, after passing over a pulley C bears a weight equal to one-third that of the beam; find the position of equilibrium.

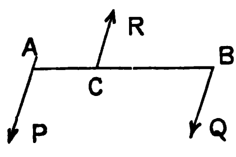
20. Weights of 40 pounds and 50 pounds are attached to A and B, the extremities of a light rigid rod AB, 17 feet long, which is supported by a cord 19 feet, tied at both ends of the beam, and passing over a small pulley O; find the position of equilibrium and the tension of the cord.

21. A uniform beam AB, of weight W, rests with one end A on a horizontal plane AC, and the other end on a plane CB, whose inclination to the horizon is 60°. If a string CA, equal to CB, prevent the beam from sliding, what is the tension?

22. To each end of a uniform straight rod, AB, 100 inches long, and weighing 12 pounds, is fastened one end of a flexible string ACB, 140 inches long, to which a weight of 9 pounds is attached at a point C, 60 inches from one end. In what position will the rod remain in equilibrium about a pivot through the middle? and where must the pivot be placed in order that the rod may be balanced when horizontal?

SECTION II.—*Couples.*

78. Introduction.—The rule in the preceding chapter for finding the resultant of two parallel forces fails when the forces are equal and opposite. For if two parallel forces P and R are opposite but not equal, R being the greater, then the resultant is (R—P), and if B be its point of application,



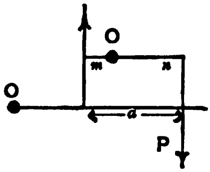
$$(R-P)\ BC = P.AC.$$

Now, if we keep P and AC the same, but make R approach P in magnitude, as (R—P) diminishes, BC increases, their product being constant, and when (R—P) becomes indefinitely small, BC must become infinitely great.

Two equal parallel forces acting in opposite directions constitute what is called a *couple.*

The perpendicular distance between the directions of the forces is called the *arm of the couple.* The tendency of the couple is to produce rotation about an axis perpendicular to the plane of the couple, and can be balanced only by another couple. It will be proved that the effect of the couple will not be altered by any change which leaves the moment about the axis the same.

79. Proposition XV.—*The moment of a couple about all points in its plane is constant.*



Take any point O in the plane of the couple PP, and let m be the distance of the point from the direction of one force, n the distance from the other. Also let a be the arm of the couple.

If the point be without the couple, the moment of the couple about the point

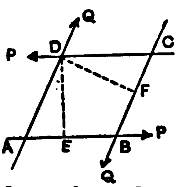
$$=P.m-P.n=P\,(m-n)=P.a.$$

If the point be between the forces, the moment

$$=P.m+P.n=P\,(m+n)=P.a.$$

80. Proposition XVI.—*Two couples with opposite rotations and equal moments will be in equilibrium.*

Let the lines of action of the forces form a parallelogram ABCD. Let the forces along AB and CD be P, and those along CB and AD be Q. Draw DE perpendicular to AB, and DF perpendicular to BC. Because the moments of the couples are equal,



$$P.DE=Q.DF\,;$$

but $AB.DE=BC.DF$;

for each product is the area of the parallelogram ABCD.

By dividing, therefore, we have

$$P : Q :: AB : BC.$$

Hence, the resultant of P and Q along AB and BC acts along the diagonal BD.

Similarly the resultant of P and Q along CD and AD acts along the diagonal DB, and is equal and opposite to the former.

Consequently, there is equilibrium.

81. Deductions.—From Propositions XV. and XVI. it is evident that—

(i.) A couple may be moved parallel to itself without altering its effect.

(ii.) Two couples are equivalent if their moments are equal and in the same direction.

(iii.) A couple may be turned in its own plane through any angle about any point in its own arm without altering its effect.

(iv.) A given couple may be replaced by another having for arm any given line in its plane.

(v.) The resultant of any number of couples acting in the same or in parallel planes is a couple, the moment of which is the algebraical sum of the moments of the couples.

(vi.) A straight line perpendicular to the plane of a couple having a length proportional to the moment of the couple is termed the *axis* of the couple; hence, a couple is determined by the direction and length of its axis.

82. Proposition XVII.—*To find the resultant of two couples not in the same plane.*

For the given couples substitute equivalent couples PP, QQ, having for common arm a portion MN of the line of

intersection of the two planes. By constructing the parallelograms, find the resultant R of the forces P and Q at M and N. Since the forces at M are equal and parallel to those at N, the resultant at M is equal and parallel to that at N, and the resultant of the two couples is a couple whose moment is $R \times MN$.

Now, since all the couples have a common arm, their moments are proportional to the lines in the parallelograms representing the forces.

If one of the parallelograms, as, for instance, that at M, be turned in its own plane about M so that each line turns through a right angle, then the sides representing P and Q will be perpendicular to the planes PP, QQ, respectively, and the diagonal representing R will be perpendicular to the plane RR. Hence the lines representing P, Q, and R, which we have seen to be proportional to the moments of the couples PP, QQ, RR, respectively, will now also have the directions of the axes of the corresponding couples.

Hence, if two straight lines, drawn from the same point, have the directions of the axes of two couples, and are proportional to the moments of the couples, that diagonal of the parallelogram on these lines which is drawn through the point has the direction of the axis and is proportional to the moment of the resultant couple.

Hence, the laws of the composition and resolution of couples are similar to the corresponding laws of forces, the axis of the couple corresponding to the direction of the force, and the moment of the couple to the magnitude of the force. For example, if L and M be the moments of the component couples, G the moment of the resultant, and θ the angle between their planes, then

$$G^2 = L^2 + M^2 + 2LM \cos\theta.$$

83. Proposition XVIII.—*Any system of forces in one plane, if not in equilibrium, is equivalent either to a single force or a couple.*

For, combine any two which do not form a couple; combine the single resultant thus obtained with a third force not forming with it a couple, and so on. The last two forces will evidently either have a single resultant or form a couple.

84. Conditions of Equilibrium of Forces in one Plane.—

(i.) If the sum of their moments about any three points not in the same straight line vanishes.

Proof.—They cannot form a couple, for then the sum of the moments about any point would be the moment of the couple. They cannot have a single resultant, for to have no moment about a point, the direction of the resultant must pass through the point, and it cannot pass through three points not in the same straight line.

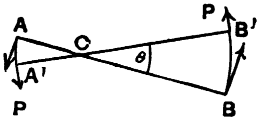
(ii.) If the sums of the moments about any two points A and B, and the sum of the components along any line not perpendicular to AB vanish.

Proof.—For the same reason as before, they cannot form a couple; and if there be a single resultant, it must act along AB. But then it would have a component along any line not perpendicular to AB, which is not the case. Therefore there is equilibrium.

(iii.) If the sum of the moments about any point, and the sums of the components in any two directions, vanish.

Proof.—Since the sum of the moments vanishes, the forces cannot be equivalent to a couple; and since there is no component force in either of two directions, there cannot be a single resultant.

85. Proposition XIX.—*To find the work done by a couple when its arm turns through an angle θ.*



Let the moment of the couple be

$$P.AB = M.$$

Let the arm turn about a point C through an angle θ.

The work done by P on the left is

$$P.AA' = P.\frac{AA'}{AC}.AC = P.\theta.AC.$$

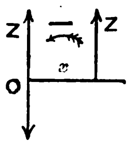
The work done by P on the right is

$$P.BB' = P.\frac{BB'}{BC}BC = P.\theta.BC.$$

Therefore the whole work is

$$P.AB.\theta = M.\theta.$$

86. Transposition of Forces.—(i.) *A force acting at one point of a body may be replaced by an equal force at any other point, and a couple in the plane of the force and point.*



Let Z be the force at A, and O any other point.

Let x be a perpendicular from O on the direction of Z.

At O add opposite forces equal and parallel to Z.

The three forces are equivalent to the first, and form a single force Z at O and a couple,—Zx.

(ii.) *A force acting at any point of a rigid body may be resolved into three components along any three lines at right angles, and three couples whose axes lie along those lines.*

Let X, Y, Z be the components of the force, x, y, z the coordinates of its point of application A, and let a plane

through A, parallel to zx, cut Oy in B. Z at A is equivalent to Z at B and couple $-\mathrm{Z}x$, Z at B is equivalent to Z at O and couple $+\mathrm{Z}y$, and similarly for X and Y.

Hence we have

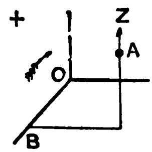
Forces at O.		Couples.		
		in xy	in yz	in xz
From Z	Z		$\mathrm{Z}y$	$-\mathrm{Z}x$
From Y	Y	$\mathrm{Y}x$	$-\mathrm{Y}z$	
From X	X	$-\mathrm{X}y$		$\mathrm{X}z$

Therefore the force P may be replaced by X, Y, Z, acting at O, and the couples of which the moments are

$$\mathrm{Z}y-\mathrm{Y}z \text{ in the plane of } (y, z),$$
$$\mathrm{X}z-\mathrm{Z}x \quad . \quad . \quad . \quad (z, x),$$
$$\mathrm{Y}x-\mathrm{X}y \quad . \quad . \quad . \quad (x, y).$$

(iii.) *To find the resultants of any number of forces acting upon a rigid body in any direction, and to find the conditions of equilibrium.*

By a resolution similar to the above of all the forces, we have them replaced by the forces



$$\Sigma\mathrm{X}, \Sigma\mathrm{Y}, \Sigma\mathrm{Z},$$

acting at O along the axes, and the couples

$$\Sigma\,(\mathrm{Z}y-\mathrm{Y}z)=\mathrm{L} \text{ suppose, in the plane of } (y, z),$$
$$\Sigma\,(\mathrm{X}z-\mathrm{Z}x)=\mathrm{M} \quad . \quad . \quad . \quad . \quad (z, x),$$
$$\Sigma\,(\mathrm{Y}x-\mathrm{X}y)=\mathrm{N} \quad . \quad . \quad . \quad . \quad (x, y).$$

Let R be the resultant of the forces which act at O; a, b, c the angles its direction makes with the axes, then

$$\mathrm{R}^2=(\Sigma\mathrm{X})^2+(\Sigma\mathrm{Y})^2+(\Sigma\mathrm{Z})^2,$$
$$\cos a=\frac{\Sigma\mathrm{X}}{\mathrm{R}}, \cos b=\frac{\Sigma\mathrm{Y}}{\mathrm{R}}, \cos c=\frac{\Sigma\mathrm{Z}}{\mathrm{R}}.$$

Let G be the moment of the couple which is the resultant of the three couples L, M, N; λ, μ, ν the angles its axis makes with the co-ordinate axes; then,

$$G^2 = L^2 + M^2 + N^2,$$

$$\cos\lambda = \frac{L}{G}, \cos\mu = \frac{M}{G}, \cos\nu = \frac{N}{G}.$$

For equilibrium $R=0$ and $G=0$.

These lead to the six conditions

$$\Sigma X=0, \ \Sigma Y=0, \ \Sigma Z=0,$$

$$\Sigma\,(Zy - Yz)=0, \ \Sigma\,(Xz - Zx)=0, \ \Sigma\,(Yx - Xy)=0;$$

or, in words, when forces act on a rigid body, in order that there may be equilibrium, the sums of the resolved parts of the forces parallel to any three lines at right angles to each other must vanish, and the sums of the moments of the forces with respect to these lines must also vanish.

EXERCISE X.

1. A triangular board is acted on by three forces along the sides and represented in direction and magnitude by the sides along which they act; show that the forces are equivalent to a couple whose moment is represented by twice the area of the triangle.

2. Prove also the analogous proposition for a polygonal body.

3. If three parallel forces are in equilibrium, they may be regarded as two couples of equal and opposite moments.

4. State and prove the three forms in which the conditions of equilibrium of a system of forces in one plane may be stated.

5. Prove that the resultant of three forces represented by LA, LB, LC, where L is the centre of perpendiculars of the triangle ABC, passes through the centre of the circle ABC.

6. A cone, whose vertical angle is 30°, and whose weight is W, is placed with its vertex on a smooth horizontal plane; show that it may be kept with its slant side in a vertical position by a couple whose arm is equal to the length of the slant side of the cone, and each force of which is $\frac{3W}{16}$.

7. A man carries a bundle at the end of a stick over his shoulder; as the portion of the stick between his shoulder and his hand is shortened, show that the pressure on his shoulder is increased. Does this change alter his pressure on the ground?

8. A man in the act of being weighed in a balance of the ordinary kind pushes with a walking-stick the beam of the balance at a point A between the point of suspension S, of the scale in which he is, and the fulcrum F. What effect, if any, will be produced on his apparent weight? If the scale in which the man is be kept from moving laterally by a horizontal string attached to a fixed point, what will be the effect?

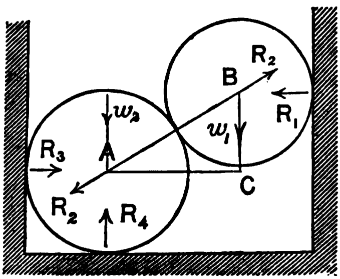
CHAPTER IX.

REACTIONS.

SECTION I.—*General Deductions from Newton's Third Law.*

87. Reactions of Surfaces.—If one surface rests against another, the reaction is along their common normal.

Example 42.—Two spheres, whose radii are respectively 3 feet and $5\frac{1}{3}$ feet, and weights w_1 and w_2, are placed between two parallel walls $16\frac{1}{3}$ feet apart, the larger sphere being the lower, and the centres being in a vertical plane perpendicular to the walls. Find the reactions, all the surfaces being smooth.



Let A and B be the centres, and let AC horizontal and BC vertical meet in C.

Then AB = the sum of the radii = 100 inches,

$$AC = \text{distance between the walls} - \text{the two radii} = 96 \text{ inches},$$

$$\therefore BC = \sqrt{(AB^2 - AC^2)} = 28 \text{ inches.}$$

Let R_1 = reaction between smaller sphere and wall,

R_2 = mutual reaction between spheres,

R_3 = reaction between larger sphere and wall,

R_4 = reaction between larger sphere and floor.

The three forces acting on B are R_1, R_2, and w_1, respectively, parallel to the three sides of $\triangle$ ABC.

$$\therefore R_1 : w_1 :: 96 : 28;$$
$$\therefore R_1 = 3\tfrac{3}{7}\, w_1,$$
$$R_2 : w_1 :: 100 : 28;$$
$$\therefore R_2 = 3\tfrac{4}{7}\, w_1.$$

By combining R_4 and w_2, the forces acting on A are reduced to three, namely, R_2, R_3, and $(R_4 - w_2)$, parallel respectively to the sides of $\triangle$ ABC.

$$\therefore R_3 : R_2 :: 96 : 100;$$
$$\therefore R_3 = \frac{24}{25} \text{ of } 3\tfrac{4}{7} w_1,$$
$$= 3\tfrac{3}{7} w_1,$$
$$(R_4 - w_2) : R_2 :: 28 : 100;$$
$$\therefore R_4 - w_2 = \frac{28}{100} \text{ of } 3\tfrac{4}{7}\, w_1,$$
$$= w_1;$$
$$\therefore R_4 = w_1 + w_2.$$

These equations may also be obtained by resolving the forces acting on each sphere horizontally and vertically.

$$\text{B}\begin{cases} \text{Vertically} & w_1 = R_2 \cos\theta = \dfrac{28}{100} R_2. \\ \text{Horizontally} & R_1 = R_2 \sin\theta = \dfrac{96}{100} \text{ of } \dfrac{100}{28} w_1. \end{cases}$$

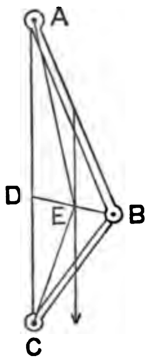
$$\text{A}\begin{cases} \text{Horizontally} & R_3 = R_2 \sin\theta = \dfrac{96}{28} w_1. \\ \text{Vertically} & R_4 - w_2 = R_2 \cos\theta = w_1. \end{cases}$$

87 *b*. If two rods cross each other, the reaction is perpendicular to both. *See* Example 36.

88. Hinges.—If two rods be hinged together, the reaction at the hinge is generally unknown, both as regards magnitude and direction.

Example 43.—AB and BC are two uniform rods, 17 inches and 18 inches long respectively. They are jointed to one another at B,

and have their other extremities attached to two pivots A and C in the same vertical line. Find the directions and magnitudes of the actions at the joints A, B, and C.



Three forces act on the rod AB, the action R at B along a line BD, the weight w_1 along the vertical line EF, and the action at A, whose direction must pass through the point E. Now the $\triangle$ ADE happens to have its sides parallel to these three forces, hence the sides are proportional to the forces, and

$$R : w_1 :: DE : AD \quad (1).$$

Similarly, by considering the forces on BC we have

$$R : w_2 :: DE : CD \quad (2).$$

But, by the Third Law, the actions at B are equal, or R has the same value in both these proportions.

$$\therefore w_1 : w_2 :: AD : CD,$$

$$\text{but } w_1 : w_2 :: AB : CB;$$

$$\therefore AD : CD :: AB : CB,$$

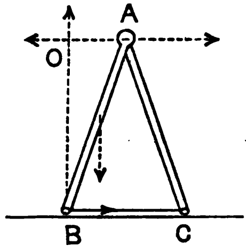
$$\therefore BD \text{ bisects } \angle ABC.$$

By adding the proportions (1) and (2) we see that if AC represents $w_1 + w_2$, BE represents R the action at B, AE the action at A, and CE the action at C.

89. Principle of Symmetry.—If there be a line through the hinge, with respect to which the figure and the forces are symmetrical, the reaction will be perpendicular to the line of symmetry. This principle of symmetry, as used in Dynamics, is a particular case of the metaphysical axiom, "Like causes under like circumstances produce like effects." In the symmetrical structures referred to, the causes and circumstances on one side of a straight line are precisely similar to those on the other; hence, so also are the effects.

Example 44.—Two equal rods AB, AC, 32·5 inches long and weighing 10 pounds, are jointed together at A; the ends B and C

are connected by a cord 40·8 inches long, and placed on a smooth horizontal plane. When the structure is at rest in a vertical plane, find the tension of the cord.



Now, the figure and the forces are symmetrical about a vertical line through A; hence, the reactions at A are symmetrical about the same line. But they are also opposite to one another (Newton's Third Law); consequently, they must be horizontal.

By taking moments about O, of the forces acting on the rod AB, as in example 37, we find $T = 4\frac{8}{253}$ lbs.

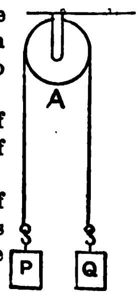
90. Tensions.—The tension of a string is the same throughout.

We may here remark that the Laws of Motion are not only to be applied where they are obviously true, or where they can be proved to be true by experiment, but they must be accepted as universally true. In the following four examples we pass gradually from cases where the truth of the law is evident to others in which it is not so.

Example 45.—Two equal weights are suspended at the end of a cord passing over a smooth pulley supported by a hook; find the pressure on the hook.

Ans.—It is evidently the sum of the weights.

Example 46.—Two unequal weights P and Q are attached to the ends of a cord passing over a smooth pulley on a hook A. When they are left free to move, find the pressure on the hook.



Let P be the mass of the larger, and Q the mass of the smaller, so that gP and gQ are the measures of the weights.

Let t be the absolute measure of the tension of the string, which is the same throughout and acts upwards on P and Q. Then evidently the pressure on A is $2t$.

Let f be the acceleration of P downwards and of Q upwards; then, as the forces acting on P and Q are respectively $gP-t$ and $t-gQ$, we have, by section 44,

$$gP-t=fP,$$
$$\text{and } t-gQ=fQ.$$

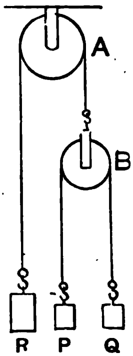
Hence, by eliminating f,

$$t=\frac{2gPQ}{P+Q}.$$

The gravitation measures of the tension and pressure on A will therefore be respectively

$$\frac{2PQ}{P+Q} \text{ and } \frac{4PQ}{P+Q}.$$

Example 47.—To one end of a cord passing over a smooth pulley fixed at A is attached a weight R, and to the other a pulley B. A cord, having at the ends unequal weights P and Q, passes over B; when the weights are left free to move, find the tensions of the cords.



Let t be the absolute measure of the tension in the cord passing over B.

Then, by the Third Law, that of the other cord is $2t$.

Let f be the acceleration of R downwards and of B upwards.

Let f' be the acceleration of P downwards, and of Q upwards, relatively to the pulley B.

Then the total acceleration of P is $f'-f$, and that of Q is $f'+f$.

The forces acting on R, P, and Q are respectively $gR-2t$, $gP-t$, and $t-gQ$.

$$\therefore gR-2t=fR \quad (1),$$
$$gP-t=(f'-f)\,P \quad (2),$$
$$t-gQ=(f'+f)\,Q \quad (3).$$

Eliminating f' from (2) and (3), we have

$$tP+tQ-2gPQ=2fPQ \quad (4).$$

Eliminating f between (1) and (4), we have

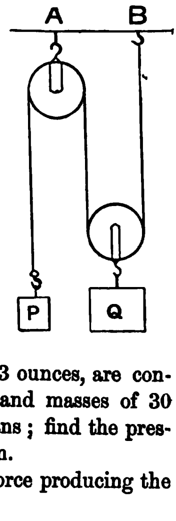
$$tPR+tQR-2gPQR=2gPQR-4tPQ;$$
$$\therefore t=\frac{4gPQR}{PR+QR+4PQ}.$$

If T be the gravitation measure of the tension of lower string, then

$$T=\frac{4PQR}{PR+QR+4PQ}.$$

The tension in the upper string is 2T, and the pressure on A, therefore, 4T.

Example 48.—A weight of 20 pounds draws up a weight of 30 pounds by a single moveable pulley, as represented in the figure; find the pressure on the supports A and B.



It is evident, from the arrangement, that if f be the acceleration of Q upwards, $2f$ is the acceleration of P downwards.

Let T be the measure of the tension of the string in pounds.

Then the force acting on P is g (P − T), and the force on Q is g (2T − Q). Hence

$$g\,(P-T)=2fP,$$
$$g\,(2T-Q)=fQ.$$

Eliminating f, therefore

$$T=\frac{3PQ}{Q+4P}=16\tfrac{4}{11}\text{ lbs.}$$

This is the pressure on B, and the pressure on A is evidently 2T or $32\tfrac{8}{11}$ pounds.

Example 49.—Two scale-pans, each weighing 3 ounces, are connected by a cord passing over a smooth pulley, and masses of 30 and 28 ounces respectively are placed in the pans; find the pressures on the scale-pans during the ensuing motion.

The whole mass moved is 64 ounces, and the force producing the motion is the weight of 2 ounces.

Hence if the mass be called 64 the force must be called $2g$, and from the formula

$$F=f.m.$$

it follows that the acceleration = 1.

The question now resolves itself into the following:—A weight of 30 ounces is acted on by an upward pressure x not sufficient to support it, but allowing it to fall with an acceleration 1; find x. Since a pressure equal to the weight of 30 ounces would prevent an acceleration of 32, whereas x prevents only an acceleration of 31;

$$\therefore \text{ as } 32 : 31 :: 30 : x;$$
$$\therefore x=29\tfrac{1}{16}.$$

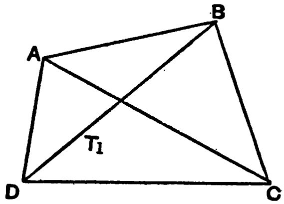
Let y be the pressure exerted by the other scale-pan on the mass of 28 ounces. Since the scale-pan not only prevents an acceleration of 32, but gives an acceleration of 1 upwards, we have

$$\text{as } 32 : 33 :: 28 : y;$$
$$\therefore y = 28\tfrac{7}{8}.$$

90 *b*. If a rod be acted on only by forces at its extremities, which, of course, excludes its having weight, the resultant forces at its extremities must balance, and therefore must act along the rod and be equal and opposite; and any portion of the rod is in equilibrium under the action of forces equal to those at the ends. If the forces are outwards, the rod is under tension, and is termed a *tie;* if they are inwards, the rod is under thrust, and is termed a *strut.*

If a string be attached to a hinge connecting two rods, its effect on each rod is indeterminate. All we know is, that the mutual actions of the rods are equal and opposite, and that the resultant of the two actions and the two stresses along the rods must be equal and opposite to the tension of the string. In dealing with the tension, therefore, we may suppose it distributed between the rods in any way whatever, provided the resultant of the supposed components is the tension. For instance, we may suppose the tension to act entirely on one rod indefinitely near the hinge, or to be equally divided between them. It must, however, be remembered that the mutual actions at the hinge will be different with every different hypothesis. It will be seen by the above section that the component of the tension on either rod, combined with the reaction on it, must give a force along the rod; hence, if there are two rods AB, AC, and a string AD, and we suppose the tension to be resolved into two components along the rods, with this supposition there will be no reactions, for if there be any, they must act along the rods, which is impossible, since they are equal and opposite.

Example 50.—Four rods jointed at their extremities, form a quadrilateral, which may be inscribed in a circle; if they be kept in equilibrium by two strings joining the opposite angular points, show that the tension of each string is inversely proportional to its length, the weights of the rods being neglected.



Resolve T_1 the tension in BD into P and Q along BA and BC.

$$\therefore P = T_1 \frac{\sin DBC}{\sin ABC};$$

Resolve T_2 into P and Q along AB and AD.

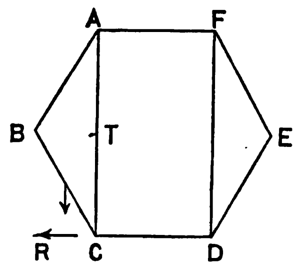
$$\therefore P = T_2 \cdot \frac{\sin DAC}{\sin DAB}.$$

$\therefore$ since $\angle DBC = \angle DAC$ and $\angle ADB = \angle ACB$.

$$\therefore \frac{T_2}{T_1} = \frac{\sin DAB}{\sin ABC} = \frac{\sin DAB \sin ACB}{\sin ADB \sin ABC}$$

$$= \frac{DB}{AB} \cdot \frac{AB}{AC} = \frac{DB}{AC}.$$

Example 51.—A regular hexagon ABCDEF is composed of six equal heavy rods moveable about their angular points; one rod AF is fixed in a horizontal position, and the ends of this rod are connected by vertical strings with the ends of the lowest rod; find the tension of each string.



Let T be the tension in either string.

Let the weight of each rod be w.

Suppose T along AC to be equal to $t + \frac{w}{2}$, the part $\frac{w}{2}$ acting on CD, and t on CB.

Then it is evident, with this supposition, that the rod CD will be supported independently of the actions on the ends; consequently, these actions must be horizontal.

Hence, R, the mutual reaction between CD and CB will be horizontal.

Take moments about B of the forces acting on CB, and let the length of each rod be a, so that $AC=a\sqrt{3}$.

$$\therefore \frac{w.a}{4}+\frac{R.a\sqrt{3}}{2}=\frac{t.a}{2};$$

$$\text{Or } w+2R.\sqrt{3}=2t.$$

Again, regard ABC as one body, and take moments about A.

$$\therefore \frac{2w.a}{4}=R.a\sqrt{3},$$

$$\text{or } w=2R\sqrt{3};$$

$$\therefore t=w;$$

$$\therefore T=\frac{3}{2}w.$$

91. Reactions in a Rigid Body.—A *body* may be considered to be an aggregation of small material particles or molecules, which are held together by their mutual attractions, or by what we may term molecular forces.

A *rigid* body is one whose molecules are held together in an invariable form, or, in a rigid body the tendency of the molecular forces to preserve the form is greater than that of the other forces which act upon the body to alter the form. The figure of a rigid body is, therefore, not affected by the forces which act upon it.

As a matter of fact, no body is perfectly rigid; but usually in machines and structures the compressibility is of insensible magnitude, and in all cases of equilibrium the body may be supposed to have assumed its figure of equilibrium, and the points of application of the forces may be regarded as a system of invariable form.

We are not acquainted with the laws according to which the molecules of a mass of matter act on each other; hence, when we have to consider the effect of forces acting on individual particles of a rigid body, or system of rigid bodies, we must look for some principle, based on the

results of experiment, which will enable us to calculate this effect without bringing the molecular forces into calculation. This principle is supplied by the First and the Third Laws of Motion. The First Law asserts that the internal reactions cannot alter the motion of a body, and therefore these reactions are in equilibrium amongst themselves; the Third Law gives us an analogous principle for any system of bodies, showing that the resultant of the actions and reactions for the whole system must be zero. For, by negation, the law implies that the action and reaction between any two bodies or particles of the system are independent of the other bodies; hence, if we consider two bodies A and B, the actions are equal and opposite; and if we add a third, C, although two new pairs of actions are introduced, the mutual action between A and B is unaltered, and so on for any number. This has been called D'Alembert's Principle.

92. D'Alembert's Principle and its Consequence.—"*The internal actions and reactions of any system in motion are in equilibrium amongst themselves.*"

If we consider one particle only of mass m, acted on by a force F and moving with acceleration f, we have

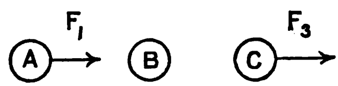
$$F = fm.$$

When this particle, however, is one of a system, there will be an action on it which must be compounded with F to give the force producing the acceleration f, so that it is no longer true either that F and fm are equal in magnitude or have the same direction. But the above principle tells us that all the actions are in equilibrium amongst themselves, so that if we combine all the forces of the system and all the products fm, then we may equate the results, leaving the actions and reactions out of all consideration, for if we could include them after the composition, they

might be removed by the principle of the superposition of equilibrium.

For the sake of clearness we will amplify this statement by applying it to the following illustrations :—

1. Let there be three particles, A, B, and C, in the same horizontal line, and let m_1, m_2, m_3, respectively, be their masses. Let forces F_1 and F_3 act on A and C, and let there be mutual attractions between A, B, and C. Now if A were the only particle, the product of the mass and acceleration of A would be equal to F_1, the force acting on A. But the acceleration of A exceeds that which F_1 would produce, for part of it is due to the attractions of B and C; similar remarks apply to B and C. Although B is not acted on by any force external to the system, it may have an acceleration in consequence of the actions of A and C on it. Now the question is, how can we find a relation between the external



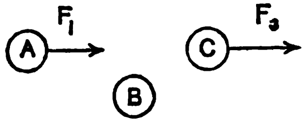
forces and the accelerations which will not involve the mutual actions? The answer is furnished by the third law of motion. Let the action of B on A be represented by + (AB), then that of A on B will be − (AB), and similar expressions apply to AC and BC. Now the whole force acting on A is F + (AB) + (AC).

$$\therefore F_1 + (AB) + (AC) = f_1 m_1.$$
$$\text{Similarly } 0 - (AB) + (BC) = f_2 m_2,$$
$$\text{and } F_3 - (AC) - (BC) = f_3 m_3;$$
$$\therefore F_1 + F_3 = f_1 m_1 + f_2 m_2 + f_3 m_3.$$

2. Let A, B, and C be in the same vertical plane, but not in the same straight line, and let F_1 and F_3 be horizontal as before.

In this case, in consequence of the attractions of B and C, the acceleration of A will neither be equal in magnitude nor the same in direction as that which would result from F_1 alone. We must here resolve the accelerations into horizontal and vertical compo-

nents; let f_x', f_x'', f_x''', be the horizontal components, and f_y', f_y'', f_y''', be the vertical components of the accelerations of A, B, and C, respectively. Then the horizontal motion is the same as if there were horizontal forces equal to $f_x'm_1$, $f_x''m_2$, $f_x'''m_3$ acting respectively on A, B, and C. But the horizontal forces are, in reality, F_1 and



F_3, together with the horizontal components of the attractions, and the latter are in equilibrium amongst themselves. Hence, when we compound all the forces the attractions disappear from the sum;

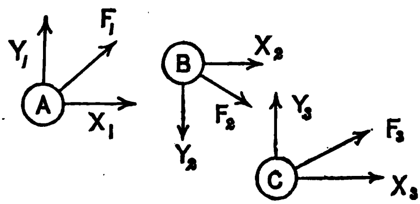
$$\therefore\ F_1+F_1=f_x'm_1+f_x''m_2+f_x'''m_3,$$

and as there are no vertical forces,

$$0=f_y'm_1+f_y''m_2+f_y'''m_3.$$

These examples show that, in the theory of this section, it is not intended that there should be external forces acting on all the particles. There may be such forces acting on a few only, or on any number of the particles. We will proceed now to more general cases, simply reminding the student that the external forces on any number of particles may be zero.

3. Let the particles A, B, C, and the forces acting on them lie in the same vertical plane, and let the horizontal components of the forces be X_1, X_2, X_3, and the vertical components Y_1, Y_2, Y_3, respectively. Let the horizontal accelerations be f_x', f_x'', f_x''', and the vertical accelerations f_y', f_y'', f_y'''.



Although we cannot say that $X_1=fm_1$, because of the actions within the system, yet if we take the sum of all the horizontal

forces, the internal actions will not appear in the sum, and we have, therefore,

$$X_1 + X_2 + X_3 = f_x'm_1 + f_x''m_2 + f_x'''m_3,$$

$$\text{or } (X_1 - f_x'm_1) + (X_2 - f_x''m_2) + (X_3 - f_x'''m_3) = 0.$$

$$\text{Similarly } (Y_1 - f_y'm_1) + (Y_2 - f_y''m_2) + (Y_3 - f_y'''m_3) = 0.$$

General Case.—Let there be any number of particles subject to their mutual actions and to external forces acting on any number of them in any directions; to find a relation between the forces and accelerations. For distinctness of reference, let us select any particle, and let the external force acting on it be F, and let the acceleration of this particle be f.

Suppose the force F and the acceleration f to be resolved into X, Y, Z, f_x, f_y, f_z, along three lines at right angles.

The motion of the body as a whole will, therefore, be the same as if all the reactions were removed and forces applied to each particle represented by the product of the mass of each particle and its acceleration. Hence, precisely the same motion may be secured either by the forces X, Y, Z at each particle, together with the reactions, or by forces equal to mf_x, mf_y, mf_z, without the reactions. Consequently, if one of these equivalent systems be reversed, so that the two become opposed, they will neutralise one another, or be in equilibrium. In other words, forces whose types are

$$X - mf_x,$$
$$Y - mf_y,$$
$$Z - mf_z,$$

applied at every particle would be in equilibrium.

The forces represented by X, Y, Z applied from without are sometimes termed the *impressed forces*, and the forces represented by mf_x, mf_y, mf_z, which would produce the actual motion without the internal actions and reactions, are termed the effective forces, and the above conclusion is

then expressed as follows:—*If forces equal and opposite to the effective forces at any instant were at that instant applied to each point of the body, they would be in equilibrium with the impressed forces.*

EXERCISE XI.

1. Two spheres, whose radii are respectively 30 and 22 inches and weights 33 and 13 ounces, are placed between two vertical rods 100 inches apart, so that their centres are in the plane of the rods and the larger sphere rests on the floor; if the surfaces are all smooth, find the reactions.

2. A smooth hemispherical shell whose base is closed includes two equal spheres whose radii are one-third of that of the shell. The shell is fixed with its base vertical; find the mutual pressures at all the points of contact.

3. Two equal rods without weight are connected at their middle points by a pin which allows free motion in a vertical plane; they stand upon a horizontal plane, and their upper extremities are connected by a thread which carries a weight. Show that the weight will rest half-way between the pin and the horizontal line joining the upper ends of the rods.

4. The whole length of each oar of a boat is 9 feet, and from the hand to the row-lock the distance is 2 feet 3 inches; each of eight men sitting in the boat pulls his oar with a force of 60 pounds. Supposing the blades of the oars not to move through the water, find the resultant force propelling the boat.

5. Two equal rods AB, AC, 5 feet long, are jointed at A and placed in a vertical plane, the ends B and C, which are connected by a cord BC, resting on a smooth horizontal plane. What must be the length of the cord in order that its tension may be two-thirds of the weight of each rod?

6. Weights of 15 and 17 ounces are connected by a cord passing over a smooth fixed pulley. When the weights are in motion find the tension of the string, taking g as 32.

7. Divide 12 pounds into two parts, so that when they are connected by a cord passing over a smooth peg and allowed to move, the tension of the cord may be $3\frac{1}{3}$ pounds.

8. If two scale-pans be connected by a cord passing over a smooth pulley, and weights of 8 ounces and 12 ounces respectively be placed in them, find the pressure on the scale-pans, supposing them to be without weight.

9. If the scale-pans in the preceding example each weigh 2 ounces, find the pressures on them.

10. In the arrangement of Example 46 find the tensions of the cords when $R=32$ ounces, $P=12$ ounces, and $Q=20$ ounces; also find how far R will move in 2 seconds.

11. If in the same figure $R=60$ and $P+Q=80$, what must P and Q be in order that R may remain at rest?

12. A weight of 12 pounds draws up a weight of 20 pounds by means of a single moveable pulley; how far will the heavier weight rise in 5 seconds?

13. If by means of a moveable block of three pulleys and a single cord a weight of 5 pounds lifts a weight of 25 pounds, find the tension of the cord and the whole pressure on the supporting beam.

14. A body P descending vertically draws another body Q up the inclined plane formed by the upper surface of a right-angled wedge which rests on a smooth horizontal table; find the force F necessary to prevent the wedge from sliding along the table.

15. The extremities of a string without weight are fastened to two equal heavy rings which slide on smooth fixed rods in the same vertical plane and equally inclined to the vertical; and to the middle point of the string a weight is fastened equal to twice the weight of each ring; find the position of equilibrium and the tension of the string.

16. Four equal and uniform heavy rods are joined by hinges so as to form a square, and two opposite angles are connected by a string; this framework stands on a fixed point, the string being horizontal; find the tension of the string.

17. A regular hexagon, composed of six equal heavy rods moveable about their angular points, is suspended from one angle, which is connected by threads with each of the opposite angles. Show that the tensions of the threads are as $\sqrt{3}:2$.

18. Four equal and uniform heavy rods are connected by hinges; the system is suspended by a string attached to one hinge, and the lowest hinge is in contact with a horizontal plane; find the tension of the string and the pressure on the plane.

19. In the arrangement represented in the figure of Example 47, page 140, R weighs 2 pounds and P and Q each weigh 1 pound. R and P are of brass and Q is of iron. A magnet placed below Q attracts it, and causes Q to move through 3 inches in 3 seconds; show that the masses R and P each rise 1 inch in this time, and find the tensions of the cords during the motion on the supposition that they are constant.

20. The stand of an Atwood's machine is placed in one scale of a balance, and is found to weigh 10 pounds. By what weight will it be balanced when 3 pounds and 5 pounds are attached to the ends of the cord respectively, and motion ensues?

21. Two scale-pans of equal weight w are connected by a fine string which passes over a smooth small pulley, and in them are placed weights w_1, w_2; show that the pressures which these weights produce on the pans during motion are

$$\frac{2w_1(w_2+w)}{w_1+w_2+2w} \text{ and } \frac{2w_2(w_1+w)}{w_1+w_2+2w}.$$

22. Show how the acceleration due to the attraction of the earth may be approximately determined by Atwood's machine. At the extremities of the cord in an Atwood's machine there are two scale-pans, each weighing 1 ounce. When 29·6 ounces are placed in one, and 32·6 ounces in the other, the scale-pans are observed to move through $\frac{5}{7}$ of a foot in a second; and when 23·785 ounces and 28·785 ounces, respectively, are placed in the pans, they move through twice this space in a second. Supposing the resistance to be the same in the two cases; find the acceleration due to gravity.

23. One end of a string is fixed to the extremity of a smooth uniform rod, and the other to a ring without weight which passes over the rod, and the string is hung over a smooth peg. The length of the string is 162 inches, 97 being on one side of the peg and 65 inches on the other; find the length of the rod and the tangent of its inclination to the horizon.

24. A smooth uniform rod AB, 5·3 feet long and weighing 20 pounds, is attached by a hinge at A to a fixed point; a cord CD,

24·7 inches long, fixed at C, is tied at D to a ring without weight which passes over the rod. The line joining CA is horizontal, and 26·5 inches in length; find the tension of the cord.

25. A heavy rod can turn freely about a fixed hinge at one extremity, and it carries a heavy ring which is attached to a fixed point in the same horizontal plane with the hinge by means of a string of length equal to the distance between the point and the hinge. Find the position in which the rod will rest.

26. Two equal beams AB, AC connected by a hinge at A are placed in a vertical plane with their extremities B and C resting on a horizontal plane; they are kept from falling by strings connecting B and C with the middle points of the opposite sides; show that the ratio of the tension of each string to the weight of each beam

$$=\tfrac{1}{8}\sqrt{(8\cot^2\theta+\operatorname{cosec}^2\theta)},$$

θ being the inclination of each beam to the horizon.

SECTION II.—*Impact.*

93. Nature of Impact.—If a body A impinges directly on a body B, momentum will be lost by A and gained by B. Now momenta lost and gained are what are termed in Newton's law action and reaction. *Hence the momentum lost during impact by one body is equal to that gained by the other, so that the whole momentum is unaltered by the impact.*

The nature of the action during impact may be thus described. When A overtakes B, so long as A moves faster than B, the surfaces will be compressed, and the compression will cease when the velocities are rendered equal; if the action stops then, the bodies are said to be *inelastic.* Let v_1 and v_2 be the velocities of two inelastic bodies A and B before impact, and V their common velocity after impact.

$$\text{The momentum before impact} = \text{A}v_1 + \text{B}v_2\,;$$
$$\text{the momentum after} = \text{AV} + \text{BV}\,;$$
$$\text{therefore } \text{A}v_1 + \text{B}v_2 = \text{AV} + \text{BV}.$$

If the velocities be not in the same direction one must be considered negative.

Generally, however, another force comes into play when the velocities are equal, and the bodies begin at that instant to recover their figure and to exert one upon the other a pressure which lasts until impact ceases. Thus A not only loses momentum during compression, but also during expansion. Let M_1 be the momentum lost by A and gained by B during compression, and M_2 that lost by A and gained by B during expansion; then it is found by experiment that the ratio $\frac{M_2}{M_1}$ is constant for the same materials. This ratio, usually denoted by e, is termed the *modulus of elasticity.*

A body is *inelastic* when $e=0$;
perfectly elastic when $e=1$;
imperfectly elastic when e is between 0 and 1.

Three assumptions therefore are made in considering impact:—

(i.) That there is an instant at which the velocities of the two bodies are equal.

(ii.) That during each of the two parts into which the period of impact is divided by this instant the action on one body is equal and opposite to the action on the other.

(iii.) That the action on each body after the instant of equal velocity is in the same direction as the action before, and that the ratio of the actions is constant.

From these assumptions, it follows that *the relative velocity after impact is* $-e$ *times the relative velocity before impact.*

Proof.—Let the masses m and m' have velocities u and u' respectively before impact, and v and v' respectively after impact. Let the period of impact be divided into two parts, called respectively the first and second, separated by the instant at which the masses have the same velocity V.

The mass m loses first momentum $m(u-\mathrm{V})$, and secondly, $m(\mathrm{V}-v)$.

The mass m' loses first $m'(u'-\mathrm{V})$, secondly $m'(\mathrm{V}-v')$.

$$\therefore\ m(\mathrm{V}-v)=em(u-\mathrm{V}),\ \text{or}\ \mathrm{V}-v=eu-e\mathrm{V},$$

$$\text{and}\ m'(\mathrm{V}-v)=em'(u'-\mathrm{V}),\ \text{or}\ \mathrm{V}-v'=eu'-e\mathrm{V}\ ;$$

$$\text{by subtraction,}\ \therefore -(v-v')=e(u-u').$$

94. Proposition XX.—*One body of mass* m *and velocity* u *impinges directly on another of mass* m′ *and velocity* u′ ; *required the velocities* v *and* v′ *after impact, the co-efficient of elasticity being* e.

The change of momentum is zero, or the momentum after impact is equal to the momentum before impact by Newton's Third Law.

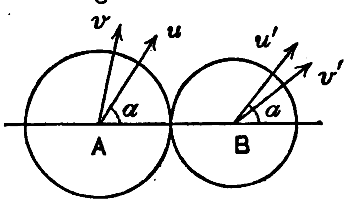
$$\therefore\ mu+m'u'=mv+m'v'.$$

The relative velocity after impact is $-e$ times the relative velocity before impact by the above assumptions.

$$\therefore\ v-v'=-e(u-u).$$

These two equations determine v and v'.

95. Oblique Collision.—Let the bodies be spherical, and let the magnitudes and directions of the velocities just before impact be u, α, u', α', and let the corresponding quantities after impact be v, β, v', β', the angles being measured from the line of centres.



As there is no force perpendicular to the line of centres,

$$v\sin\beta=u\sin\alpha,$$

$$\text{and}\ v'\sin\beta=u'\sin\alpha'.$$

Considering the motion along the line of centres, the fact that the change of momentum is zero gives

$$mu\cos\alpha+m'u'\cos\alpha'=mv\cos\beta+m'v'\cos\beta'\ ;$$

and the fact that the relative velocity after impact is equal to $-e$ times the relative velocity before impact, gives

$$v\cos\beta - v'\cos\beta' = -e(u\cos\alpha - u'\cos\alpha').$$

Example 48.—A body weighing 5 pounds moving with a velocity of 14 feet per second impinges on a body weighing 3 pounds, and moving with a velocity of 8 feet per second; the elasticity being one-third; find the velocities after impact.

Momentum before impact = momentum after,

$$\therefore 5\times 14 + 3\times 8 = 5v + 3v'.$$

Relative velocity after $= -e$ times relative velocity before,

$$\therefore v - v' = -\tfrac{1}{3}(14-8) = 2,$$

$$\therefore v' = 13 \text{ and } v = 11.$$

If a body impinge on a smooth fixed plane, the velocities along the plane before and after impact are the same, and the velocity perpendicular to the plane after impact is numerically e times the velocity perpendicular to the plane before impact.

97. Proposition XXI.—*By the collision of two imperfectly elastic bodies the kinetic energy of the system is diminished.*[1]

(i.) *Direct collision.*—Let u and u' be the velocities before impact of two bodies whose masses are m and m' respectively, and v and v' their velocities after impact. Then

$$mv + m'v' = mu + m'u', \text{ and } v - v' = -e(u-u').$$

Therefore $(mv+m'v')^2 = (mu+m'u')^2$,

and $mm'(v-v')^2 = mm'e^2(u-u')^2$;

therefore by addition, $(m+m')(mv^2+m'v'^2)$

$$= (m+m')(mu^2+m'u'^2) - mm'(1-e^2)(u-u')^2;$$

therefore,

$$mv^2 + m'v'^2 = mu^2 + m'u'^2 - \frac{mm'}{m+m'}(1-e^2)(u-u')^2.$$

[1] The bodies are supposed to recover their form after impact, and to have in each case no other motion than the direct motion of the body as a whole. The proposition will be discussed more fully, with the view of determining what becomes of the lost kinetic energy, in the volume on heat.

Now e cannot be greater than unity, so that $1-e^2$ cannot be negative;

hence $mv^2+m'v'^2$ is always less than $mu^2+m'u'^2$, except when $e=1$, and then the two expressions are equal.

(ii.) *Oblique collision.*—Let a and a' be the angles which their directions of motion make with the line of centres before impact, β and β' the corresponding angles after impact.

Then,

$$(mv\cos\beta+m'v'\cos\beta')^2=(mu\cos a+m'u'\cos a')^2,$$

$$\text{and } mm'(v\cos\beta-v'\cos\beta')^2=mm'e^2(u\cos a-u'\cos a')^2$$

$$=mm'(u\cos a-u'\cos a)^2-mm'(1-e)(u\cos a-u'\cos a')^2.$$

Hence, by addition, and by division by $m+m,'$

$$mv^2\cos^2\beta+m'v'^2\cos^2\beta'$$

$$=mu^2\cos^2 a+m'u'^2\cos^2 a'-\frac{mm'(1-e^2)}{m+m'}(u\cos a-u'\cos a')^2.$$

$$\text{Also } mv^2\sin^2\beta=mu^2\sin^2 a,$$

$$\text{and } m'v'^2\sin^2\beta'=m'u'^2\sin^2 a.$$

Therefore, by addition,

$$mv^2+m'v'^2=mu^2+m'u'^2-\frac{mm'(1-e^2)}{m+m'}(u\cos a-u'\cos a')^2;$$

and as $1-e^2$ cannot be negative, the required result is obtained.

If the elasticity be perfect, the kinetic energy of the system is the same after the collision as before.

EXERCISE XII.

1. An inelastic body moving with velocity v impinges on another of twice its mass at rest; find the velocity after impact.

2. The weights of A and B are 6 pounds and 10 pounds, and they move in the same direction with velocities of 8 feet and 6 feet per second; required their velocities after impact—1st, when the bodies are inelastic; 2d, when they are perfectly elastic; 3d, when the coefficient of elasticity is one-third.

3. Two inelastic bodies, A and B, weigh 12 pounds and 7 pounds respectively, and move in the same direction with velocities of 8 feet and 5 feet in a second; find their common velocity after impact; also the velocity lost by A and that gained by B respectively.

4. A and B are perfectly elastic, and A, with a velocity of 20 feet per second, strikes B at rest; find their velocities after impact —1st, when $A=B$; 2d, when $A=4B$.

5. The weights of A and B are 3 pounds and 5 pounds, and their velocities are 7 feet and 9 feet per second, in opposite directions; required their velocities after impact, in the same cases as in the first Example.

6. A, moving with a velocity of 11 feet, impinges upon B, moving in the opposite direction with a velocity of 5 feet, and by the collision A loses one-third of its momentum; what are the relative magnitudes of A and B?

7. A, weighing 8 pounds, impinges upon B, weighing 5 pounds, and moving in A's direction with a velocity of 9 feet in 1 second; by collision, B's velocity is trebled; what was A's velocity before impact?

8. A is equal to 3B, and impinges upon B at rest; A's velocity after impact is two-thirds of its velocity before impact; required the value of e, which measures the elasticity.

9. Find the elasticity of two bodies, A and B, and their proportion to each other, so that when A impinges upon B at rest, A may remain at rest after impact, and B move on with one-sixth of A's velocity before impact.

10. At what angle must a body, whose elasticity is one-third, be incident on a plane, that the angle between the directions before and after impact may be a right angle?

CHAPTER X.

CENTRE OF GRAVITY.

98. Introduction.—The attraction of the earth, which causes a body to have weight, acts on every particle of the body; if, like a stone, for example, it can be broken into small fragments, the sum of the weights of the small particles will be equal to that of the whole body.

If one of these particles be attached by a fine thread to a fixed point, the thread will take the direction of the vertical through the point, and if several of them be suspended from points near together, the threads will be parallel. When, therefore, the particles are united so as to form the body, we may regard their weights as a system of parallel forces.

It is true that, regarding the earth as a sphere, the vertical lines would all converge to the centre, and therefore, strictly speaking, the directions of the forces which the earth exerts on the different particles composing a body are not parallel. But since the dimensions of any body we have to consider are very small compared with the radius of the earth, we may consider these directions to be appreciably parallel, and, therefore, the resultant attraction on the body or system to be equal to the sum of the attractions on the constituent particles; *i.e.* the *weight* of the whole equal to the sum of the weights of the several parts.

We may, therefore, at once apply to the weights of any

system of particles the theorems respecting parallel forces given in Chapter VII.

Let us suspend a body by means of a fine cord attached to a point A in it; the tension of the cord will be a force in equilibrium with the resultant of the weights of the particles which compose the body; hence, this resultant will be equal and opposite to the tension, and will therefore have the direction of the vertical through A. Suspend the body again from another point; the weight of each particle will have the same magnitude and the same point of application as before, but the direction of the forces with regard to the body will be changed. The result is the same as if each force had been turned about its point of application; hence, the new line of support or direction of the resultant will intersect the old one in the centre of the forces. If the body were composed of a plastic material, and pierced in the direction of the line of support in several different positions, all the lines of perforation would intersect in the same point. This point is termed the *centre of gravity* of the body. We are led, therefore, to the following definition :—

99. Definition of Centre of Gravity.—*The resultant of a system of parallel forces acting on a rigid body passes through a fixed point, the position of which is independent of the direction of the forces. If the forces be the weights of the several elements of the body, the fixed point is termed the centre of gravity.*

If the resultant of the forces acting on a body be equal to the weight of the body, and act vertically upwards through the centre of gravity, it is evident from the definition that the body will be at rest. This statement is sometimes taken as the definition of the centre of gravity, and expressed thus :—

The centre of gravity of a system of heavy particles is a point such that, if it be supported and the particles rigidly connected with it, the system will rest in any position.

The process of finding the centre of gravity of a number of heavy particles is precisely the same as that of finding the centre of a system of parallel forces.

100. Volumes and Surfaces.—We shall be concerned only with bodies through the volumes of which matter is distributed uniformly. Such bodies are termed homogeneous. A solid body is homogeneous when any two parts of equal volume are exactly of the same weight. The determination of the centre of gravity of a homogeneous body is, therefore, a purely geometrical question. The weights of different portions will be proportional to their volumes; hence, the volumes may represent the forces.

Again, consider a very thin sheet of metal or paper of uniform thickness. The weights of any two portions will be proportional to the areas; hence, we may treat the areas as forces, and are then said to find the centre of gravity of the *surface*.

In like manner, if we take a very thin wire of uniform thickness, the weights of different portions will be proportional to their lengths. We may therefore find the centre of gravity of a heavy line.

101. Symmetrical Figures.—The following considerations will assist us in solving problems connected with the centre of gravity:—

(i.) If a body be symmetrical about a plane, the C.G. lies in that plane. Every particle on one side corresponds to an equal particle on the other. Hence, the C.G. of every pair of particles lies in the plane, and, therefore, so also does the C.G. of the whole.

(ii.) It follows that if a body have two planes of symmetry, the C.G. lies in their line of intersection; and if it have three planes of symmetry intersecting in two lines, the C.G. is at the point where the lines cut one another.

(iii.) If an area be symmetrical about a line, the C.G. lies in that line.

(iv.) If a body have a centre of figure, that is, a point such that all lines drawn through it to the outline of the figure are bisected in the point, the centre of figure is the C.G.

For any two lines drawn through this point will contain figures on opposite sides of the vertex, in every respect equal. If any point in one of these figures be joined with the corresponding point in the other, the line drawn will be bisected by the centre; therefore the line joining their C.G.s will be bisected by the centre of figure. Consequently, this point is the C.G. of the two figures; similarly it is the C.G. of every other pair included by lines through the centre, and therefore it is the C.G. of the whole.

From these facts we may conclude at once that—

1. The C.G. of a straight line is its middle point.
2. The C.G. of the circumference or area of a circle is the centre.
3. The C.G. of the perimeter or area of a parallelogram is the point of intersection of the two diagonals, for this point is the centre of figure.
4. The C.G. of the volume or surface of a sphere is the centre.
5. The C.G. of a right circular cylinder is the middle point of the axis.
6. The C.G. of a parallelopiped is the point of intersection of two diagonals.
7. The C.G. of a regular figure coincides with the C.G. of its perimeter, and the C.G. of equal weights at its angular points.

102. Application of Theory of Parallel Forces.—By applying the theorems on parallel forces to the particular

case when the forces are weights, we have at once the following rules and propositions :—

(i.) The C.G. of two particles at points A and B, whose masses are m_1, m_2, is at a point C in AB, such that

$$\frac{m_1}{\text{CB}}=\frac{m_2}{\text{AC}}=\frac{m_1+m_2}{\text{AB}}. \quad \text{(Prop. XI.)}$$

(ii.) The C.G. of a number of particles lying on a straight line will be found by multiplying the mass of each by its distance from a fixed point in the line, adding the results and dividing by the sum of the masses. The quotient is the distance of the C.G. from the fixed point (Prop. XII.)

(iii.) If a number of particles lie in the same plane, and a fixed straight line be taken in the plane, the sum of the products of each mass by its distance from the line is equal to the distance of the C.G. from the same line multiplied by the sum of the masses. If we take two lines at right angles therefore, to determine $\bar{x}$, $\bar{y}$ the distances of the C.G. from them, we have the equations

$$\Sigma mx=\bar{x}.\Sigma m \text{ and } my=\bar{y}.\Sigma m. \quad \text{(Prop. XIII.)}$$

(iv.) If any number of particles be taken and the mass of each be multiplied by its distance from a fixed plane, the sum of the products equals the distance of the C.G. from the plane multiplied by the sum of the masses. If, therefore, we take three planes at right angles, we obtain three equations—

$$\Sigma mx=\bar{x}.\Sigma m,$$
$$\Sigma my=\bar{y}.\Sigma m,$$
$$\Sigma mz=\bar{z}.\Sigma m,$$

which determine the position of the C.G.

(v.) Given the centres of gravity of two parts which compose a body, to find the centre of gravity of the whole body.

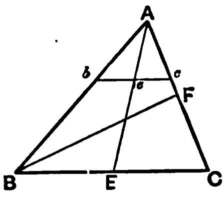
Let G_1 denote the centre of gravity of one part, and G_2 the centre of gravity of the other part; let m_1 denote the mass of the first part and m_2 the mass of the second part. Join G_1, G_2, and divide it in G, so that $\frac{GG_1}{GG_2}=\frac{m_2}{m_1}$; then G is the centre of gravity of the whole body.

(vi.) Given the centre of gravity of a body, and also the centre of gravity of a part of the body, to find the centre of gravity of the remainder.

Let G denote the centre of gravity of the body, and G_1 the centre of gravity of a part of the body; let m denote the mass of the body, and m_1 the mass of the part. Join G_1G, and produce it through G to G_2, so that $\frac{GG_2}{GG_1}=\frac{m_1}{m-m_1}$; then G_2 is the centre of gravity of the remainder.

103. Proposition XXII.—*To find the C.G. of a plane triangle.*

Let ABC be the triangle; bisect BC in E; join AE; draw any straight line ceb parallel to CEB, cutting AE in e. Then, by similar triangles,



$$ce : CE :: Ae : AE,$$
$$\text{and } be : BE :: Ae : AE,$$
$$\therefore ce : CE :: be : BE,$$
$$\text{but } CE=BE, \therefore ce=be.$$

Hence, AE bisects every line parallel to BC. Therefore each of the strips similar to ceb, into which we may suppose the triangle to be divided, has its C.G. on AE, and therefore the centre of gravity of the whole must be in the line AE.

Similarly we may show that the C.G. lies in any middle line of the triangle.

But it is proved by geometry that the three middle lines of a triangle pass through the same point and trisect one

another; hence, the C.G. of a plane triangle may be found either by drawing two middle lines and taking their point of intersection, or by drawing one middle line and marking off a point in it one-third of its length from the base.

104. Proposition XXIII.—*To find the C.G. of three equal particles placed at the angles* ABC *of a triangle.*

The resultant of P at B and P at C will be 2P at E and the C.G. of 2P at E, and P at A is at a point G, such that EG equals one-half AG. Hence, the C.G. of the three equal particles coincides with the C.G. of the triangle.

105. Proposition XXIV.—*To find the C.G. of the perimeter of a triangle.*

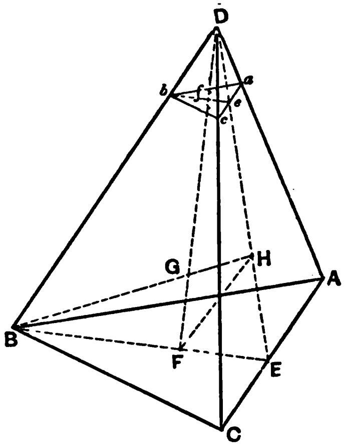
The C.G. of each side is at its middle point. Hence, we have to find the C.G. of forces acting at the middle points A′, B′, C′ of the sides BC, CA, and AB, respectively, and being proportional to the lengths of the sides. Join the points A′, B′, C′. The C.G. of the forces at C′ and B′ is at a point D, such that

$$\frac{DB'}{DC'}=\frac{AB}{AC}=\frac{\frac{1}{2}AB}{\frac{1}{2}AC}=\frac{A'B'}{A'C'}.$$

Hence, the C.G. of the whole perimeter is in the line A′D. Since the line A′D divides the base C′B′ into parts proportional to the sides, it bisects the angle A′. Similarly we may show that the C.G. lies in the line bisecting the angle B′. Now, the intersection of the bisectors of two angles of a triangle is the centre of the inscribed circle; hence the C.G. of the perimeter of a triangle coincides with the centre of the circle inscribed in the triangle, whose angular points bisect the sides of the original triangle.

106. Proposition XXV.—*To find the C.G. of a pyramid on a triangular base.*

Let ABC be the base, D the vertex; bisect AC in E; join BE, DE; take EF$=\frac{1}{3}$EB, then F is the centre of gravity of the base ABC. Join FD; draw *ab*, *bc*, *ca* parallel to AB, BC, CA, respectively, and let DF meet the plane *abc* in *f*; join *bf* and produce it to meet DE in *e*. Then, by similar triangles, *ae*=*ec*; also



$$\frac{bf}{\mathrm{BF}}=\frac{\mathrm{D}f}{\mathrm{DF}}=\frac{ef}{\mathrm{EF}};$$

but $\mathrm{EF}=\frac{1}{2}\,\mathrm{BF}$,

$$\therefore\ ef=\tfrac{1}{2}\,bf;$$

therefore *f* is the C.G. of the triangle *abc*; and if we suppose the pyramid to be made up of an indefinitely great number of indefinitely thin triangular slices parallel to the base, each of these slices has its C.G. in DF. Hence, the C.G. of the pyramid is in DF.

Again, take EH$=\frac{1}{3}$ED; join HB, cutting DF in G. Then, as before, the C.G. of the pyramid must be in BH; hence, it must be G, the point of intersection of the lines DF and BH.

Join FH; then FH is parallel to DB.

Also because EF$=\frac{1}{3}$EB, therefore FH$=\frac{1}{3}$DB, and

$$\frac{\mathrm{FG}}{\mathrm{FH}}=\frac{\mathrm{DG}}{\mathrm{DB}};\ \text{but } \mathrm{FH}=\tfrac{1}{3}\mathrm{DB},\ \therefore\ \mathrm{FG}=\tfrac{1}{3}\mathrm{DG}=\tfrac{1}{4}\mathrm{DF}.$$

Hence, the C.G. is in the line joining the C.G. of the base with the vertex at one-fourth of its length from the base.

107. Proposition XXVI.—*To find the C.G. of a pyramid with polygonal base.*

Decompose the solid into triangular pyramids by planes passing through one edge AB. Take a point a in AB such that $Ba = \frac{1}{4}AB$. A plane through this point parallel to the base will cut the pyramid in a section similar to the base, and will divide all lines from A to the base in the same proportion. Hence, the C.G. of each pyramid lies in this section. Therefore the C.G. of the whole pyramid lies in this plane.

Again, join the vertex with the C.G. of the base, then every section parallel to the base will be similar to the base; and if we suppose the pyramid divided into thin slices by planes parallel to the base, the C.G. of each slice will be on the line joining the vertex with the C.G. of the base. Hence, the C.G. of the whole lies in this line; and consequently it is the point where this line cuts the plane drawn at one-fourth of the height of the pyramid parallel to the base. Hence the C.G. of every pyramid is in the line joining the C.G. of the base with the apex at one-fourth of its length from the base.

We may consider a cone as a pyramid with a great number of faces; hence, the C.G. of a cone is in the line joining the C.G. of the base with the apex at one-fourth of its length from the base.

108. Proposition XXVII.—*The work done in raising a number of weights through different heights is equal to the work which would be done if all the weights were concentrated at their C.G. and raised from the first position of the C.G. to its second position.*

Take a fixed plane lower than the lowest of the weights; consider one body of weight W, and let it be raised from a height x_1, to a height x_2, above the fixed plane; then the work done is

$$w\,(x_2 - x_1);$$

similar expressions give the works in the case of the other particles.

Hence, the whole work done is

$$\Sigma\,(\mathrm{w}x_2-\mathrm{w}x_1),$$
$$\text{or } \Sigma\mathrm{w}x_2-\Sigma\mathrm{w}x_1.$$

But by Prop. XIII. if $\bar{x}_1$, $\bar{x}_2$ be the first and last heights of the C.G. of all the weights,

$$\Sigma\mathrm{w}x_1=\bar{x}_1.\Sigma\mathrm{w}\,;\ \Sigma\mathrm{w}x_2=\bar{x}_2.\Sigma\mathrm{w}.$$
$$\therefore\ \Sigma\mathrm{w}x_2-\Sigma\mathrm{w}x_1=(\bar{x}_2-\bar{x}_1).\Sigma\mathrm{w}.$$

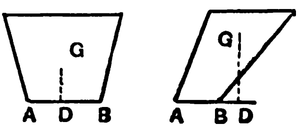
But $(\bar{x}_2-\bar{x}_1)$ is the height through which the C.G. is raised, and $\Sigma\mathrm{w}$ is the whole weight. Hence, the proposition is proved.

109. **Proposition XXVIII.**—*A body placed on a horizontal plane will stand or fall according as a vertical line drawn through its C.G. falls within or without the base.*

The only force acting on the body besides the resistance of the plane is its own weight, and this acts in a vertical direction through the C.G.

Let the area which would be enclosed by a string drawn tightly round the points of support be termed the BASE.

Let G be the C.G. of the body. Let the vertical line through G cut the horizontal plane on which the body stands at D. Let any horizontal straight line be drawn through D, and let AB be that portion of it which is within the base of the body.



First suppose D to be *between* A and B.

No motion can take place round A. For the weight of the body acts vertically downwards at G, and has a moment W.AD tending to turn G round A in the direction GD, which is prevented by the resistance of the plane.

Similarly no motion can take place round B. The same reasoning applies to any other horizontal line through D and terminated by the base. Therefore the body cannot fall.

Next, suppose D *not* to be *between* A and B; let it be on AB produced through B.

Then motion will take place round B in consequence of the moment W.BD, as there is nothing to prevent this motion.

110. Kinds of Equilibrium.—If a body at rest under the action of forces cannot move without its C.G. being raised, its equilibrium is *stable*. For example, a cube resting on a rough plane is in stable equilibrium.

If, when the body is moved, the position of its C.G. must be lowered, the equilibrium is *unstable*. For example, a cone resting on its vertex is in unstable equilibrium.

If when a body is moved, the C.G. remains in the same horizontal plane, the equilibrium is *neutral*; a sphere resting on a plane is in neutral equilibrium.

In the case of *stable* equilibrium the work done by a small disturbing force will be spent in lifting the weight and the body will first come to rest, and then in consequence of the energy stored in it will return to its original position.

In the case of *unstable* equilibrium, on the action of the lightest force the body will receive kinetic energy in the direction of motion (in consequence of the work done by the fall of the C.G. of the body), which will carry it away from its first position. Hence, the equilibrium is stable, when the body will return to its original position if slightly displaced; unstable, when the body will fall away from its original position if slightly displaced. Consequently, every body suspended from a point is in stable equilibrium, and every body supported above a point or line is in unstable equilibrium.

Example 52.—A triangle is immersed in water, so that the depths of its angular points are respectively 12, 7, and 8 inches; find the depth of its C.G.

The C.G. coincides with that of three equal weights of one pound at the angular points; by § 102, iv.,

$$1 \times 12 + 1 \times 8 + 1 \times 7 = 3 \times x.$$

$$\therefore x = 9 \text{ inches.}$$

Example 53.—Find the C.G. of a conical shell contained between two right circular conical surfaces, having the same axis, the outer diameter of the shell being 8 inches, the inner diameter 6 inches, and the height of the whole 12 inches.

Since the heights are proportional to the diameters of the bases, the height of the cavity is to 12 as 6 to 8; hence it is equal to 9 inches.

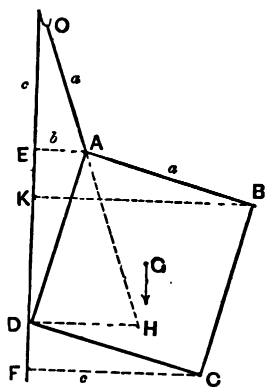
The volume of a cone the height of which is h, and the radius of base r is $\pi.r^2\frac{h}{3}$.

Hence the volume of the solid contained by the outer conical surface $=\pi.64$, and that contained by the inner surface $=\pi.27$; therefore the volume of the shell $=\pi.37$. The C.G. of the whole cone is 3 inches from the base, and that of the cavity is $2\frac{1}{4}$ inches from the base.

$$\text{Hence } \pi.64 \times 3 = \pi.27 \times 2\tfrac{1}{4} + \pi.37 \times x.$$

Therefore $x = 3\frac{81}{148}$ = the distance of the C.G. of the shell from the base.

Example 54.—A square rests with its plane perpendicular to a smooth wall, one corner being attached to a point in the wall by a string whose length is equal to a side of the square; show that the distances of three of its angular points from the wall are as 1, 3, and 4.



Let ABCD be the square, G its C.G., and OA the string. Draw perpendiculars to the wall from A, B, C, and D.

Let $OA=a$, $OE=c$, $AE=b$; then the following right-angled triangles are in all respects equal, namely, those which have for hypothenuse OA, AD, AB, CD.

Hence $OD=2c$, $\therefore DH=2b$, $CF=c$ $BK=b+c$.

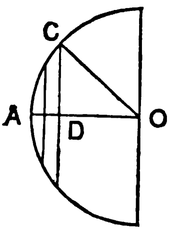
Now, if equal weights were placed at A, B, C, and D, their C.G. would be that of the square. This fact gives the following equation by § 102:—

$$1 \times AE + 1 \times BK + 1 \times CF + 1 \times 0 = 4 \times DH,$$
$$\text{or } b + (b + c) + c = 8b,$$
$$\therefore c = 3b,$$
$$\text{that is, } FC = 3AE;$$
$$\text{and } b + c = 4b,$$
$$\text{or } BK = 4AE.$$

Example 55.—The equations in § 102 (iv.) may be used to find the C.G. of a body which can be divided into parts, the C.G.'s of which can be determined. We will take as an example the following exercise:—

To find the C.G. of a *hemisphere.*

Let O be the centre and A the vertex of the hemisphere.



From the symmetry of the figure, it is evident that the C.G. is in OA. Divide the radius, OA into n equal parts: then if the length of the radius be a, that of each part will be $\frac{a}{n}$.

Take sections of the hemisphere perpendicular to AO, through two adjacent points of division, one being at D, distant x divisions from O.

If the divisions be sufficiently small, we may regard the enclosed volume as that of a circular plate, of volume $\pi . CD^2 \frac{a}{n}$, the distance of its centre from O being $x . \frac{a}{n}$.

The product of the volume by distance from O is

$$\therefore \pi CD^2 \frac{a^2}{n^2} x = \frac{\pi a^2}{n^2}\left(a^2 x - x^3 \frac{a^2}{n^2}\right).$$

Now if we give x, in succession, all values from 1 to n and sum the results, we shall have a quantity equal, by the proposition, to

the whole volume multiplied by the distance of the C.G. from $O\bar{x}$, that is, to $\frac{2}{3}\pi a^3\bar{x}$;

$$\therefore \frac{\pi a^4}{n^2}(1+2+\ .\ .\ .\ +n)$$

$$-\frac{\pi a^4}{n^4}(1^3+2^3+\ .\ .\ .\ +n^3)=\tfrac{2}{3}\pi a^3\bar{x}.$$

The first series is an arithmetical progression, whose sum is $\frac{1}{2}n$ $(n+1)$. The second reduces to the square of the same, since the sum of the cubes of consecutive numbers, commencing with unity, is equal to the square of the sum.

$\therefore$ we have,

$$\frac{a}{n^2}(\tfrac{1}{2}n.\overline{n+1})-\frac{a}{n^4}(\tfrac{1}{4}n^2\overline{n+1}^{\,2})=\bar{x}\tfrac{2}{3};$$

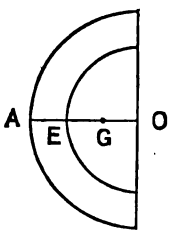
$$\therefore a\left(\tfrac{1}{2}.\overline{1+\frac{1}{n}}\right)-a\left(\tfrac{1}{4}.\overline{1+\frac{1}{n}}^{\,2}\right)=\tfrac{2}{3}\bar{x}.$$

Now when n is made very great, the quantity $\frac{1}{n}$ vanishes, and then

$$\tfrac{1}{2}a-\tfrac{1}{4}a=\tfrac{2}{3}\bar{x};$$
$$\therefore \bar{x}=\tfrac{3}{8}a.$$

Example 56.—From this we may deduce the position of the C.G. of a hemispherical shell.

Suppose the whole hemisphere divided into pyramids, with small bases on the surface, and with their vertices at O.



The C.G.'s of all will lie on a hemispherical surface, having centre O and radius OE $=\frac{3}{4}$OA (§ 107). The surfaces intercepted on the inner hemisphere are proportional to the volumes of the pyramids, and therefore the C.G. of the hemispherical surface with radius OE must coincide with that of the solid hemisphere, with radius OA. Hence it must be at G, which is $\frac{3}{8}$ of OA or $\frac{3}{8}$ of $\frac{4}{3}$ of OE $=\frac{1}{2}$OE.

Hence the C.G. of a hemispherical shell is at a distance of one-half of the radius from the centre.

EXERCISE XIII.

1. Find the C.G. of two solid bodies of 12 pounds and 20 pounds respectively, the line joining their C.G.s being 2 feet 8 inches.

2. The distance of the C.G. of two heavy particles from the greater is 5 inches, and the particles are respectively 60 and 72 grammes; find the distance between them.

3. Weights of 2 pounds, 4 pounds, 6 pounds, and 8 pounds are placed so that their C.G.s are in a straight line, and 6 inches apart; find the distance of their common C.G. from that of the largest weight.

4. If three men support a triangular board at its three corners, what portion of the weight will each bear?

5. A figure is formed of a square, each side of which is 9 inches, and an isosceles triangle, equal in area to half the square on one of the sides; find the distance of the C.G. of the whole from the common side.

6. The middle point of one side of a square is joined with the middle points of the adjacent sides, and the triangles thus formed are cut off; find the C.G. of the remainder.

7. Two spheres touch one another; find the distance of their C.G. from the point of contact, the radii being respectively 8 inches and 12 inches.[1]

8. Find the C.G. of weights of 3, 4, 5, and 6 pounds, placed at the corners of a square, the side of which is 1 foot.

9. A chain 100 feet long and weighing 2 pounds to the foot is suspended by a link 10 feet from one end, on a peg 20 feet high, so that 70 feet are coiled on the ground. Find the work required to haul the chain to the level of the peg.

10. A ladder 25 feet long, weighing 60 pounds, and having the C.G. 5 feet from one end, stands against a vertical wall with the foot 15 feet from the wall; find the measure of the work done in lifting it on to a horizontal floor 10 feet above its top.

11. Find the C.G. of a uniform circular disc out of which another

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[1] The volume of a sphere$=\frac{4}{3}\pi r^3$ ($\frac{22}{7}$ is an approximation to π), and the volumes of spheres are as the cubes of their radii.

circular disc has been cut, the diameter of the latter being equal to the radius of the former.

12. Find the C.G. of a heavy bar, 10 feet long, bent so as to form an angle of 60 degrees, 4 feet from one end.

13. If ABC be a right-angled triangle, and p, q, and r weights at the angles, find the distance of the C.G. of the weights from the angle A, having given AB$=80$ inches, BC$=39$ inches, CA$=89$ inches, $p=15$ pounds, $q=14$ pounds, $r=12$ pounds.

14. If two triangles stand on the same base, the line joining their C.G.s is parallel to the line joining their vertices.

15. A circular table, weight W, is supported on four legs in the circumference which form a square; find the least weight which will overturn the table when placed at the circumference.

16. A frame in the form of a regular hexagon, weighing a ton, has to be raised from a horizontal position, so that its angular points are in order at the following heights, 12, 10, 7, 6, 8, and 11 feet; find the measure of the work to be done.

17. A rod 5 feet long is supported by a cord 7 feet long, tied to the two ends and passing over a smooth peg. It is found that the rod rests when the parts of the cord between the peg and the ends of the rod are respectively 3 feet and 4 feet long; find the position of the C.G. of the rod.

18. Two uniform rods forming a right angle are suspended from the end of the shorter. If one be 18 inches and the other 24 inches long, find the inclination a of the shorter to the horizon.

19. In the above, if the shorter rod$=2m$, and the longer $2n$, show that

$$\tan a=\frac{m^2+2mn}{n^2}.$$

20. Find the position of the centre of gravity of four uniform rods forming a trapezoid, the two parallel sides of which are respectively 12 inches and 30 inches long, and the other sides are each 15 inches long.

21. Find the position of the C.G. of the space enclosed by the four rods.

22. Find the C.G. of a triangle, and prove that it coincides with that of three equal particles at its angular points.

AD, BE, CF are the perpendiculars from the angles of a triangle ABC on the opposite sides: prove that the C.G. of six particles at A, B, C, D, E, F proportional to $\sin^2A$, $\sin^2B$, $\sin^2C$, $\cos^2A$, $\cos^2B$, $\cos^2C$, respectively, coincides with that of the triangle DEF.

23. A uniform wire is bent into the form of three sides AB, BC, CD of an equilateral polygon, and its C.G. is at the intersection of AC, BD; show that the polygon must be a regular hexagon.

24. A triangle ABC is successively suspended from the angles A and B, and the two positions of any side are at right angles to each other; show that

$$5c^2 = a^2 + b^2.$$

25. A triangular plate hangs by three parallel threads attached at the corners, and supports a heavy particle. Prove that if the threads are of equal strength, a heavier particle may be supported at the C.G. than at any other point of the disc. (Refer to Example 39.)

26. If a cone have its base united concentrically to the base of a hemisphere of equal radius, find the height of the cone that the solid may rest on a horizontal table on any point of its spherical surface.

27. The sides of a heavy triangle are 3, 4, 5, respectively: if it be suspended from the centre of the inscribed circle show that it will rest with the shortest side horizontal.

28. The altitude of a right cone is h, and the diameter of the base is b; a string is fastened to the vertex and to a point on the circumference of the circular base, and is then put over a smooth peg: show that if the cone rests with its axis horizontal the length of the string is $\sqrt{(h^2+b^2)}$.

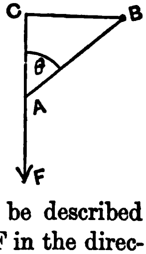
29. If a uniform wire be bent into the form of a triangle, and at the middle points of the sides there be placed three beads whose weights are proportional to the sides on which they are; prove that when the beads are moved with equal velocities in the same direction along the sides, there will be no change in the position of the C.G. of the whole system.

CHAPTER XI.

ENERGY.—*Continued.*

111. Introduction.—In this chapter we intend to consider those extensions to the subject of Chapter III., namely, the energy of a moving particle, which follow from the Laws of Motion and, in particular, to deduce the Theory of Machines. It will be proved that the relation between work and kinetic energy established for a single particle is true for any system of particles or any machine whatever, acted on by such forces as are found in Nature. The chapter will conclude with an explanation of the term Potential Energy, and an exposition of its properties.

112. Recapitulation of Theory of Work.—When a particle moves in the direction of a force acting on it, the work done by the force is the product of the force and the distance through which the particle moves; when the direction of motion of the point of application of the force is not the direction of the force, the work done by the force, while the point moves from a position A to another position B, may be described either as the work done by the component of F in the direction AB, or the work done by F while its point of applica-



tion moves through a distance AC, equal to the projection of AB on the direction of the force.

From the first, work$=$F $\cos\theta$.AB,

From the second, work$=$F.AB $\cos\theta$,

and these products are evidently identical.

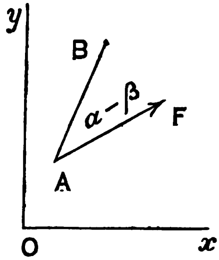
It has been shown that when a material particle is acted on by a constant force, the variation of the kinetic energy (or the product of the square of the velocity and half the mass) in a given time is equal to the work done by the force in that time. This variation is an increase when the force acts in the direction of the initial velocity, and a decrease in the opposite case; in other words, the change in the product $\frac{1}{2}v^2m$ is positive or negative, according as the force is a moving or a resisting force.

By representing the work done by an area, we remove the restriction as to the constancy of the force; and by taking for the work of a force, that of its component in the direction of the line joining the two positions of the particle, we remove the restriction as to direction.

It must, however, be distinctly noticed that we have not removed all restrictions as to the kinds of force to which the theory of energy applies; for our rule for finding the work done by a variable force is to use exactly the same processes as are employed to find the area bounded by a curve; hence the theory applies only to such forces as can be represented by the ordinates of a curve, the abscissæ of which represent the spaces passed over by the points of application. That it may be possible so to represent the law of variation, it is evident that the forces must depend only on the positions of the points of application. Now all the forces known in Nature fulfil this condition, since they consist of mutual reactions of bodies, varying only with their distances, or forces tending to fixed centres and depending only on the distances from these centres. For

example, we could not apply the theory to forces depending both on the positions and directions of motion of the points of application; no such forces, however, are met with in Nature.

113. Analytical Expressions for the Work of a Force.—Let the components of a constant force F parallel to two straight lines or axes Ox, Oy at right angles be X and Y, and let the direction of F make with the axis of x the angle α so that



$$F\cos\alpha = X,\ F\sin\alpha = Y;$$

let A and B be two positions of the point of application of F, and let AB make with the axis of x an angle β, and therefore with the direction of F an angle $(\alpha-\beta)$, and let the projections of AB on the axes be x and y, so that

$$AB\cos\beta = x,\ AB\sin\beta = y.$$

The work done by F while its point of application moves from A to B is

$$F\cos(\alpha-\beta)AB,$$

$$\text{or } F\cos\alpha.AB\cos\beta + F\sin\alpha.AB\sin\beta,$$

$$\text{or } Xx + Yy, \quad . \quad . \quad . \quad (a).$$

That is to say, the whole work done by a force is equal to the *sum* of the works done by the rectangular components of the force, the work of each being estimated in its own direction.

Similarly when dealing with motion in space of three dimensions, the work done by a force, F, while its point of application moves from A to B, the line AB making an angle θ with the direction of the force is $F(AB)\cos\theta$. But

if the direction of F makes angles α, β, γ with the axes, and AB makes angles α', β', γ', then

$$\cos\theta = \cos\alpha \cos\alpha' + \cos\beta \cos\beta' + \cos\gamma \cos\gamma'.$$

Hence, the components being X, Y, Z, and projections of AB x, y, z, the work is

$$\mathrm{F}\cos\theta(\mathrm{AB}) = \mathrm{F}\cos\alpha(\mathrm{AB})\cos\alpha' + \mathrm{F}\cos\beta(\mathrm{AB})\cos\beta' + \mathrm{F}\cos\gamma(\mathrm{AB})\cos\gamma' = \mathrm{X}x + \mathrm{Y}y + \mathrm{Z}z.$$

If the force be not constant from A to B, we can represent each of these terms by an area (§ 53), or divide the path into parts so small, that we may suppose the force constant through each of them. Then, if x', y', z' be projections of a small portion of the path, the work done will be

$$\Sigma(\mathrm{X}x' + \mathrm{Y}y' + \mathrm{Z}z') \quad . \quad . \quad . \quad (\alpha)$$

the summation extending along the whole path, from A to B. We will represent this sum for one particle by w_{ab}, in other words this symbol represents the work done while the particle moves from A to B. Hence, the equation of § 53 may be written,

$$w_{ab} = \tfrac{1}{2}mv_b^2 - \tfrac{1}{2}mv_a^2.$$

If the point A be not the origin from which x, y, z are measured, and if the co-ordinates of it be x_a, y_a, z_a, and those of B, x_b, y_b, z_b, then the work done between A and B is evidently the work from the origin to B minus that from the origin to A, and the equation may be written thus—

$$w_b - w_a = \tfrac{1}{2}m\,(v_b^2 - v_a^2).$$

The above equation (α) suggests another difference between momentum and energy; to compound momenta we must apply the method of the Parallelogram of Forces, to compound energies we simply add them. Relative directions have to be considered in the composition of momenta, but not in the composition of energies. We will illustrate this difference in an example.

Example.—A body projected up a rough inclined plane rising 3 in a length of 500, the friction being a fiftieth of the weight, comes to rest after moving 2000 feet; find the time of motion.

Let us apply § 56 and § 50.

If v be the initial velocity, and $\frac{W}{g}$ the mass of the body, the work accumulated in the beginning is $\frac{Wv^2}{2g}$.

The work done against friction is $2000W \div 50$, or $40W$.

As the body is lifted vertically through a height of 12 feet, the work done against the weight is $12W$.

$$\text{Hence } \frac{Wv^2}{2g} = 40W + 12W.$$

$$\therefore v = \sqrt{(64 \times 52)} = 16\sqrt{13}.$$

$$\text{Again, the accumulated momentum is } \frac{W}{g}v.$$

We cannot do with the products Ft as with the products Fs, taking each in its own direction and adding the results; we must find the sum of the components in the direction of the plane, namely, $\frac{W}{50}$ due to friction and $W \sin\alpha$, or $\frac{3W}{500}$, the resolved part of the weight in the direction of the plane.

$$\therefore \frac{W}{g}v = \left(\frac{W}{50} + \frac{3W}{500}\right)t.$$

$$\therefore t = 69 \text{ seconds.}$$

114. Forces acting on a Rigid Body or System of Bodies.—Let us take a system of bodies under the action of forces. We will call the forces applied from without, as distinguished from the internal actions and reactions of the system, the *external forces.* In the case of a rigid body, or a system of bodies, it is not necessarily true of any particle that the work done by the external forces acting on it is equal to the gain of kinetic energy of the particle, for there are mutual actions between it and the other particles of the system which may also do work, and the gain of kinetic

energy is equal to the work done by the external forces, plus the work done by the actions on the particle which arise within the system.

Let us consider the work done by these forces while the system passes from one position which we may term the *first* configuration to another position which we may term the *second* configuration, and let any position from which the system may be supposed to start be called the initial configuration. Let the positions of any particle P in the first and second configurations, respectively, be A and B. Let v_a and v_b be the velocities of P at A and B, and let w_a, w_b be the works of the forces acting on P while it moves from its initial position to the points A and B, respectively, then we have already seen that

$$w_b - w_a = \tfrac{1}{2}(mv_b^2 - mv_a^2).$$

Now let similar equations be formed for the other particles of the system and the whole be added together. Using the symbol Σ to indicate as usual the sum of quantities of a given type, the result may be written

$$\Sigma(w_b - w_a) = \tfrac{1}{2}\Sigma(mv_b^2 - mv_a^2).$$

For all cases in which the work done by the internal actions during the time considered is zero, the w refers only to the work of external forces, and the equation

$$\Sigma(w_b - w_a) = \tfrac{1}{2}\Sigma(mv_b^2 - mv_a^2)$$

is simply the algebraical expression of the statement that the sum of the works of the external forces in the given time is equal to the gains of kinetic energy in the same time. This statement holds good, for instance (1.), when the actions occur only at fixed points; (2.) when the actions are always perpendicular to the direction of motion as when they keep the particles moving in circles; (3.) when the icles are rigidly connected or move so that the lines

joining them are of constant length; (4.) when at the beginning and the end of the time considered the particles occupy exactly the same positions.

115. Deductions from the Equation of Energy.—When the right-hand side of the preceding equation is zero, the left must be zero, and *vice versâ*. By examining the circumstances under which these modifications in the equation of energy occur, we may make the following deductions:—

(i.) If the forces vary only according to such laws as admit of the application of the rule for determining the total work done (§ 53), in every system of bodies and in every machine which has arrived at a state of uniform motion, since the kinetic energy of the system is invariable, the sum of the works of the forces is zero during any time. In this case the work of the moving forces must be equal and opposite to the work of the resisting forces.

For example: when a train moves with uniform speed, the force exerted by the locomotive is exactly equal to the resistances of the air and friction. If when a weight is being raised by a crane, all the wheels of the machine turn with uniform velocities, then the work done by the engine is exactly equal and opposite to the sum of the works of the resistances of the air and friction, and the work of the weight; in other words, while the above conditions hold, the sum of *all* the works is zero.

(ii.) If the bodies of the system, or the particles of the machine have velocities which vary with the time, but return periodically to the same values, then the sum of the works is not zero for every portion of time, but it is zero for any complete period, or any number of whole periods.

For example: consider a pendulum oscillating under the action of gravity in a resisting medium such as the air; between two instants at which the velocity of the bob is zero, the whole work done is zero; that is to say, the work

done by gravity downwards equals the sum of the work done against gravity upwards, and the work done against the resistance of the air.

(iii.) When the forces acting on the system are such as admit of the application of the rules for finding the work done in any time, if in any series of successive transformations the bodies or particles of the system are found twice in the same situations, and if at the two epochs the velocity of any particle has the same value, so that the sum of the kinetic energies is the same, then the sum of the works of the forces for the interval which separates these epochs is zero.

(iv.) It is impossible by any combination to obtain a machine such that when the parts are put in motion and left to their mutual reactions, and to the action of weight or analogous forces, they shall return ultimately to their original position, either with increased velocities, or with the original velocities, having in the meantime done work. For, if we consider a complete cycle or period separating two instants of identical position since the points of application of the reactions and weights have reached their original position, the whole work done by each and all is zero. Hence, even if there are no resistances the kinetic energy cannot have increased, and if work has been done to overcome resistances, it must have been done at the expense of the accumulated energy; in other words, the velocities must have decreased.

No machine can be made either perfectly rigid, or so that the parts will move upon one another entirely without friction. Hence, no machine can be devised which, when left to itself, will move for ever. This fact is sometimes expressed by saying that *perpetual motion is impossible.*

(v.) Any force which acts upon a fixed point of the system, will not appear in the equation of energy, since the velo-'es of the point are nothing. In this way the mutual

pressures of any parts of the system against immoveable obstacles will not appear; neither will the force of friction which acts upon a body rolling upon a fixed obstacle, since the point of contact is for the instant at rest.

If there is friction between bodies sliding one upon the other, the friction enters into the equation.

(vi.) If there are no external forces, the sum of the changes of kinetic energy is zero; hence, the whole kinetic energy is constant throughout the motion,

$$\text{or, } \Sigma \text{MV}^2 = \text{C}.$$

Conversely, if in any system the sum of the changes of kinetic energy is always zero, no external forces act on the system.

If, therefore, a change takes place in the velocity of one body of the system, there must be a corresponding change in the velocity of some other body or bodies. These changes would arise from their mutual actions having done work. If the work of the actions of the rest of the system on one body be zero during any time, then its velocity will be the same at the end as at the beginning of the time. Hence, it follows that if all the bodies of a system be found twice in the same positions, moving in the same directions, there being no external forces, the velocities at the two instants must be the same.

(vii.) If at any given instant the external forces acting on a rigid body are in equilibrium, then at that instant the kinetic energy of the body will not be undergoing change. For a very small interval of time, therefore, the sum of the works of the forces must be zero. As the spaces passed through in this small interval by the points of application of the forces will be proportional to the velocities of the points, we may make this statement and its converse in the following terms:—

If each force be multiplied by the velocity of its point of

application in its own direction, when the forces are in equilibrium, the sum of the products will be zero; and, conversely, when the sum of the products is zero, the forces are in equilibrium.

(viii.) By the second law of motion, when forces act on a body at rest they would be in equilibrium if the body were moving through the position of rest; hence, if the forces are in equilibrium for a given position of the body, they will also be in equilibrium if the body move in any way so that each point passes through the position of rest at the instant under consideration.

We may, therefore, imagine the body to move from its position of rest in any manner we please, and conclude, that for a very small displacement the sum of the works of the forces will be zero. In this case the relative velocities are virtual or imaginary, and the principle is therefore termed the Principle of Virtual Velocities. It is usually stated as follows:—

If any machine is in equilibrium under the action of a system of forces, and we conceive any small displacement of the machine consistent with the connection of its various parts, the algebraical sum of the virtual works of the forces is zero; and, conversely, if the sum be zero for all displacements the forces are in equilibrium.

116. Potential Energy.—Suppose a body to be moving in a straight line under the action of a force which, if variable, depends only on the position of the body. Let its distances from a fixed point in the line at two instants of time be a and b, and its velocities at the two instants v_a, v_b; also let w_a, w_b be the works done by the force while the body moves from the fixed point over the distances a, b, respectively; then the work done between the points is

$$(w_b - w_a);$$

$$\therefore w_b - w_a = \tfrac{1}{2}(mv_b^2 - mv_a^2).$$

Now let C be the greatest amount of work that can be done by the force on the particle, commencing at the fixed point or origin, then C, the whole work possible at the beginning, exceeds w, the work done since the beginning, by a quantity which we will denote by P.;

$$\therefore w_a = C - P_a,$$
$$\therefore w_b = C - P_b,$$
$$\therefore w_b - w_a = P_a - P_b,$$
$$\therefore P_a - P_b = \tfrac{1}{2}mv_b^2 - \tfrac{1}{2}mv_a^2,$$
$$\therefore P_a + \tfrac{1}{2}mv_a^2 = P_b + \tfrac{1}{2}mv_b^2.$$

Consequently the quantity $P + \frac{1}{2}mv^2$ is constant.

The term $\frac{1}{2}mv^2$ is, as we have seen (§ 53), equal to the work required to produce the velocity v, and also to the work that could be done by the body before that velocity becomes zero; hence, it expresses power of doing work, and has been rightly termed *energy*. The same name, however, may be given to the quantity represented by P, for it stands for the amount of work which may yet be done by the force. The quantity $\frac{1}{2}mv^2$ has been termed *kinetic*[1] *energy* because it is the power of doing work which the body has *in virtue of its motion.*

The greater the amount of work done, the greater the kinetic energy, but the less P, the amount of work which can yet be done. P is, therefore, *energy in store.* If the force depends only on the distance from a fixed point, P depends only on the position of the body; hence, it may be termed energy of position. It has also been aptly called possible or *potential* energy, because it represents the power the body has of acquiring actual or kinetic energy, if it be allowed to yield to the forces which act upon it, and this is the term we will adopt.

The energy of a moving body is therefore of two kinds,

[1] (Gr.) *kineo*, I move; *kinetos*, moving.

potential and kinetic; and the sum of the two kinds is constant.

117. Extension to any number of Bodies.—These remarks apply to any particle or to any system of particles. For let $W_b - W_a$ stand for $\Sigma w_b - \Sigma w_a$, the motion being either in one plane, or in space of three dimensions, then

$$W_b - W_a = \tfrac{1}{2}\Sigma m v_b^2 - \tfrac{1}{2}\Sigma m v_a^2.$$

Now let C be the greatest amount of work which can possibly be done by all the forces, when their points of application move from the origin, and P_a, P_b the greatest amounts that can be done, commencing at two other positions or configurations of the system, respectively, then

$$C - W_a = P_a,$$
$$C - W_b = P_b;$$
$$\therefore W_b - W_a = P_a - P_b = \tfrac{1}{2}\Sigma m v_b^2 - \tfrac{1}{2}\Sigma m v_a^2,$$
$$\therefore P_a + \tfrac{1}{2}\Sigma m v_a^2 = P_b + \tfrac{1}{2}\Sigma m v_b^2.$$

Hence, the sum of the potential and kinetic energies of the system is constant.

118. Illustrations of change of Potential Energy.—i. *Falling Bodies.*—The simplest illustration is the case of a falling body. When a body is thrown vertically npwards, its actual energy diminishes as it rises; and when the body reaches the highest point of its course, its actual energy is spent. The body is not, however, in the same condition as at starting. If free to fall to its first position, it will acquire an actual energy exactly equal to that which has been expended in raising it. Thus the energy that was given to it has not been lost, but has been converted into an *advantage of position*, or *Potential Energy.*

For instance, if a body weighing 1 pound be projected with a velocity which would carry it vertically to a height of 100 feet, when it starts there will be 100 units of work

in it; when it has passed through 60 feet, there will be only 40 units of work accumulated in it. But the body being 60 feet higher than before, will have gained an advantage of position, represented by 60 units; thus 60 units of actual energy have been changed to potential energy, and at any instant of its flight its actual energy plus its potential energy will be equal to the whole energy with which it started.

When it reaches the highest point of its course, the stone's energy is entirely potential: when it reaches the ground, its energy is again entirely kinetic, and in any intermediate position, its energy is partly potential and partly kinetic, the sum of the two kinds being constant.

ii. *Attractions.*—Let two bodies at positions A and B at a distance d attract one another with a force varying directly as their masses m_1 and m_2, and inversely as the square of the distance between them, so that the force of attraction is proportional to the quantity $\frac{m_1 m_2}{d_2}$.

Suppose A fixed and B moveable. Let K be the kinetic and P the potential energy of B, then

$$\text{K}+\text{P}=\text{C} \text{ (a constant).}$$

At the instant at which the velocity of B is zero K is zero, and all the energy is potential. Hence C is the potential energy when the body B is at rest.

When B moves up to A, P becomes zero, and all the energy is kinetic. Hence C is the kinetic energy which B would have if it moved up to A.

The influence of the mass m_1 at the fixed point A will pervade space, so that every point of space is affected by its existence. If there be a point such that without the existence of the mass at A a particle of matter placed at the point will be at rest, then, in consequence of the existence

of the mass at A, the particle will move when placed at the same point. Now, it is convenient to have some means of estimating the influence at different points of space of a body or a system of bodies, whether there is matter at the given point or not, and this we are able to do by means of a quantity termed the Potential, due to A at the given point. We proceed to explain this term.

Potential.—If a small body of mass m_2 moves from an infinite distance up to a given point under the action of the attraction of a body A, it will be acted on at every point of its course by a force which, since it has the mass m_2 for a factor, may be represented by the product m_2R, where R is a quantity which varies from point to point of the path. The whole work done by the attraction as the body moves from an infinite distance to the given point will also have m_2 for a factor, for if we divide the path into very small portions, the work in each small portion will have the force for a factor. Hence the work up to any point may be represented by the product m_2V. Then the quantity V is termed the *potential* at the given point due to the mass A.

If m_2 is now made the unit of mass, V is the work done by the attraction while this mass moves from an infinite distance up to the given point, hence we may define the potential as follows :—

The potential at any point due to any attracting body is the quantity of work which will be done by the mutual attraction between the body and a unit of mass while this unit of mass moves from an infinite distance up to the point.

This is the definition of the potential as it is generally used in the theory of gravitation. It will be seen that it is here a quantity of the opposite sign to potential energy. The difference of the potentials at two points due to any body is equal to the difference of the two values of the

mutual potential energy between the body and a unit of mass placed at the points, but has the opposite sign.

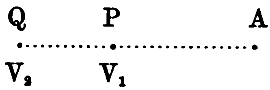
In the theories of Magnetism and Electricity, the same sign is given to the potential and the potential energy. The latter is then defined thus—

The potential at any point due to any attracting or repelling body is the quantity of work required to remove a unit of matter from that point to an infinite distance.

In this case, therefore, the difference of the potentials at two points equals the difference of the values of the mutual potential energy between the body and a unit of matter placed at the two points.

The difference of potential between two points is, therefore, the work required to carry an independent unit of matter from one point to the other.

Relation of Potential and Force.



Let A be the attracting body, and let us use the second definition of potential.

Let P and Q be points distant x_1 and x_2, respectively, from A, and let V_1, V_2 be the potentials at these points.

The work required to move an independent unit of matter from P to Q is therefore $V_1 - V_2$. Now, suppose P and Q to be so near that the attraction on the unit of matter at these points in the direction QP is the same and equal to F. The work done *against* F while the unit of matter moves from P to Q will therefore be $-F(PQ)$;

$$\therefore \; -F(PQ) = V_1 - V_2.$$

Let $V_1 - V_2$, the difference of potential between P and Q, be called V', and let $(x_1 - x_2)$ be called x', so that $-PQ = x'$;

$$\therefore \; F = \frac{V'}{x'}.$$

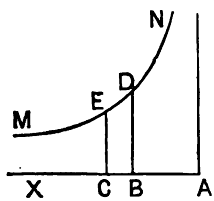
Hence the force in the direction of the line joining two adjacent points is the difference of the potentials at the points divided by the distance between them; in other words, the rate of *decrease* of the potential per unit of length at any point is the measure of the force at that point.

If we had used the first definition, we should, of course, have obtained the same result with opposite sign, so that the force at a given point in any direction would be measured by the rate of *increase* of the potential at that point.

If two adjacent points have the same potential, it is evident that there is no force in the direction of the line joining them.

Graphic representation of Potential.

Let m be the attracting mass at a point A. Let B and C be points in the line AX drawn from A towards the left. Let MN be a curve, such that the ordinate BD at any point B in AX represents the force on a unit of mass at the point B. Then the potential at B is the whole area on the left of BD, between the curve DM and the line BX. The difference of potential between two points, B and C, is the area BDEC, or area BDEC$=V_1-V_2$.



Let the law of force be that the force varies inversely as the square of the distance. Let AB$=x$, and BC$=d$, and suppose d small. Then the area BDEC evidently lies between CE.CB and BD.CB, that is, between $\frac{md}{(x+d)^2}$ and $\frac{md}{x^2}$. Since d is small, we may take for the area the geo-

metrical mean of these quantities $\frac{md}{x(x+d)}$, which is equal to $\frac{m}{x}-\frac{m}{x+d}$.

$$\therefore V_1-V_2=\frac{m}{x}-\frac{m}{x+d}.$$

Hence, if V be the potential at any point distant r from A

$$V=\frac{m}{r}.$$

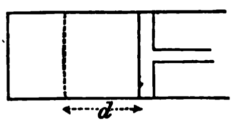
CHAPTER XII.

MISCELLANEOUS PROBLEMS ILLUSTRATING THE THEORY OF ENERGY.

Introduction.—In this chapter a number of important problems will be demonstrated by means of the principles explained in the last chapter, and will serve as illustrations of the relations between the two kinds of energy. The earlier problems on the external work done by a body expanding under pressure lie at the base of the theory of heat-engines, and the last three problems contain the theory of the Simple Pendulum.

(i.) *A piston is pressed into a cylinder containing a fluid which exerts a constant pressure on the piston of* p *pounds per unit of area. If* v *be the volume of the space through which the piston moves, and the weight of the piston be independently supported, the work done is* p.v.

Let a be the area of the piston. The whole pressure on the piston$=p.a.$ Let this pressure be extended through a distance $d.$



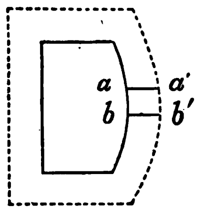
Then the work done$=pad.$

But ad=the volume, v, of the space through which the piston moves.

Hence the work$=p.v.$

(2.) *The surface of a body is subject to a constant pressure of* p *pounds per unit area; if the body expands so that its volume increases by a small quantity* v′, *then the work done in the expansion is* p.v′.

Take a small portion, *ab*, of the surface of the body, and suppose it to take the position *a′b′* very near to *ab* after expansion.



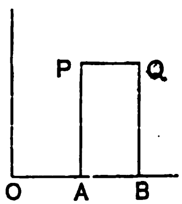
Then, if *m* be the volume of the space through which *ab* moves during the expansion, the work done against the pressure it sustains is *pm*.

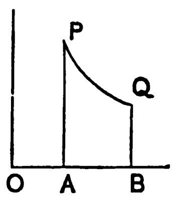
The same reasoning applies to all the small portions into which the surface may be divided.

Hence, by taking the sum of these quantities for the whole surface, it follows that the whole work done is *p.v′*.

(3.) *To represent graphically the work done by the expansion of a body against a pressure tending to resist the expansion.*

Let OA and OB represent the volumes of the body before



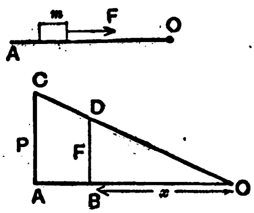


and after expansion, respectively, and let PQ be a line such that the ordinate at any point M represents the pressure corresponding to the volume OM. Let PA, QB be the initial and final pressures. If the pressure be uniform, the

figure ABQP will be a rectangle, and by the preceding proposition the area ABQP will represent the work done. If the pressure be not uniform, PQ will be a curve, and by a process similar to that used in § 11 and § 52, we may prove that in this case also the area ABPQ represents the work done.

(4.) *A body of mass* m *is placed on a smooth horizontal plane at a point* A, *and is acted on by a force tending to a fixed point* O, *and not constant, but always proportional to the distance of the body from the fixed point* O. *Determine the nature of the motion, and the velocity at any point.*

Let P be the force at the point A, let the distance AO$=a$.



At A, draw AC to represent P, and join CO. Let B be any other point in AO at distance x from O, let F be the force, and let BD be the ordinate at B.

Since BD : AC : : $x : a$,

and F : P : : $x : a$;

$$\therefore F = P\frac{x}{a},$$

and is represented by BD.

The work done by the force while the body moves from A to O is represented by the area AOC; hence, if V be the velocity at O,

$$\tfrac{1}{2}Pa = \tfrac{1}{2}mV^2;$$

$$\therefore P = \frac{mV^2}{a} \text{ and } F = \frac{mx}{a}V^2.$$

Let v be the velocity at B, then since the potential energy at B is represented by the area BDO, or $\frac{1}{2}Fx$,

$$\therefore \tfrac{1}{2}Fx + \tfrac{1}{2}mv^2 = \text{a constant}.$$

To determine this constant, it may be remarked that

when $x=0$, $v=V$;

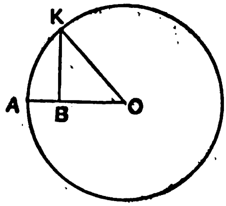
$$\therefore \tfrac{1}{2}Fx+\tfrac{1}{2}mv^2=\tfrac{1}{2}mV^2, \text{ or substituting for } F;$$

$$\therefore \frac{mx^2V^2}{a^2}+mv^2=mV^2;$$

$$\therefore v^2=V^2\left(\frac{a^2-x^2}{a^2}\right).$$

(5.) *To find, in the preceding example, the time of a complete oscillation.*

With O as centre and OA as radius, describe a circle. At B draw BK perpendicular to AO and join OK. We will now show that as the point B which marks the position of the body at any instant moves along AO, the point K moves on the circumference with constant velocity.



The velocity at B is given by the equation

$$v^2=V^2\left(\frac{a^2-x^2}{a^2}\right)$$

$$=\frac{V^2}{a^2}\cdot BK^2$$

$$v=\frac{V}{a}BK.$$

Now the triangle OKB has the side OK perpendicular to the velocity of K, OB perpendicular to the vertical component, and BK perpendicular to the horizontal component of this velocity. Moreover, the horizontal component, which is evidently the velocity of the body at B, is proportional to BK; hence, by the triangle of velocities the actual velocity of K is proportional to OK, and

$$=\frac{V}{a}\cdot OK=V.$$

The angular velocity of K is therefore $\frac{V}{a}$. Hence if T be the time of a complete oscillation of B, and therefore a complete revolution of K,

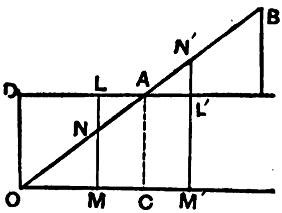
$$T \cdot \frac{V}{a} = 2\pi;$$

$$\therefore T = \frac{2a\pi}{V} = 2\pi\sqrt{\frac{am}{P}}.$$

(5.) *A body of mass* m *is suspended by an elastic string, the tension exerted by which varies in the same proportion as the length stretched. The body is at first supported so that the string is vertical and has its natural length, and then the external support is suddenly removed. Determine the nature of the motion and the velocity at any point.*

At any instant the body is acted on by two forces, T the tension of the string, and W the weight.

Draw Ox a horizontal straight line on which to set off the spaces passed over by the body, or which is the same thing, the stretched lengths of the string.



From O draw OB so that the ordinate at any point represents the tension when the string is stretched to that point.

Let OD represent the weight of the body, and draw DA horizontal, then at any point M, ML represents the weight, MN the tension, and therefore LN the force acting on the body.

The work done while the body moves from O to M= area ODLN;

$$\therefore \text{area ODLN} = \tfrac{1}{2}mv^2.$$

Let V be the velocity at C, the point at which the tension equals the weight, and let $OC = a$

$$\triangle OAD = \tfrac{1}{2} AC.OC = \tfrac{1}{2} W.a;$$
$$\therefore \tfrac{1}{2} Wa = \tfrac{1}{2} mV^2.$$

At C the body has a velocity, in consequence of which the string continues to stretch, and at a point M′ beyond C, T is greater than W or $M'N' > M'L'$. The tension is therefore destroying the velocity of the body, and at last it is brought to rest for an instant. If BE represent the force at this instant, the area BAE = area OAD, and AE = AD.

Hence we see that when the weight is applied suddenly the string will be stretched twice as much as when applied gradually, and the tension will be doubled.

The time of a complete oscillation is by the preceding exercise $\dfrac{2a\pi}{V}$,

$$\text{or since } \tfrac{1}{2} W.a \text{ or } \tfrac{1}{2} gma = \tfrac{1}{2} mV^2;$$
$$\therefore V^2 = ag;$$
$$\therefore T = 2\pi\sqrt{\frac{a}{g}}.$$

(7.) *A particle of mass* m *describes a circle of radius* A *with uniform velocity* v *under the action of a force tending to a centre* O, *find the relation between the force and the velocity.*

It is evident that the force is constant, for at every point all other circumstances of the motion are the same.

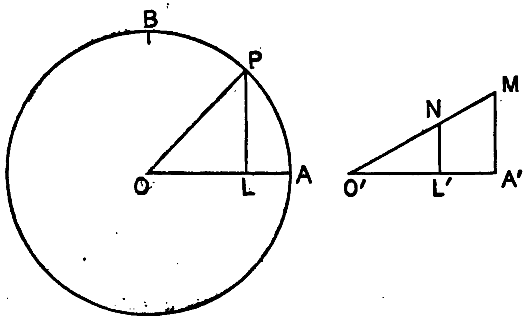
Let the radius OA be horizontal, and let the particle move from A towards B. Let us resolve the force and velocity through the quadrant AB into horizontal and vertical components, and equate the work effected and the kinetic energy accumulated in one of these directions.

The horizontal velocity at A is zero, and that at B is v, hence the gain of horizontal kinetic energy is $\frac{1}{2}mv^2$.

The whole force F is horizontal at A, and at B the horizontal compound of the force is zero. At any point P in AB, if PL be perpendicular to OA, and θ be the angle subtended at the centre by AP, the horizontal component of the force at P

$$=F\cos\theta=\frac{F.OL}{OP}=\frac{F.OL}{OA}.$$

To calculate the work done, draw an auxiliary figure thus; let O′A′=OA, O′L′=OL, let A′M perpendicular



to O′A′ represent F and join MO′. Then the ordinate L′N : A′M : : O′L′ : O′A′ : : OL : OA. Hence L′N represents the horizontal component of the force at the corresponding point P. Consequently the area of the triangle O′A′M represents the horizontal work done, therefore this work is $\frac{1}{2}a$F;

$$\therefore \tfrac{1}{2}aF=\tfrac{1}{2}mv^2.$$

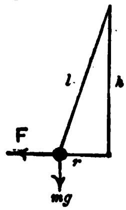
Let f be the normal acceleration, so that $F=fm$;

$$\therefore fm=\frac{mv^2}{a};$$

$$\therefore f=\frac{v^2}{a}.$$

(8.) *The Conical Pendulum.—A heavy particle is suspended from a fixed point by a fine thread, and is set in motion in such a way that it describes a small horizontal circle, find the time of a whole revolution.*

Let the mass be m, and weight therefore mg, and let v be the velocity of the particle, then by symmetry v is constant.



Let the length of the thread be l, the radius of the circle r, and the height from the centre of the circle to the fixed point h.

The particle is acted on by three forces, the tension of the string along l, the weight mg vertical, and a force F horizontal, and directed from the centre.

This force produces the normal acceleration

$$\frac{v^2}{r} \text{ (by preceding example);}$$

hence if ω be the angular velocity,

$$F = \frac{mv^2}{r} = m\omega^2 r.$$

Now, as the particle has no velocity in the plane of these forces, the forces must be in equilibrium, and as they are parallel to the sides of the triangle, they are proportional to the sides.

$$\therefore F : mg :: r : h,$$

$$\text{or } m\omega^2 r : mg :: r : h;$$

$$\therefore \omega = \sqrt{\frac{g}{h}}.$$

But if t be the time of a complete revolution,

$$t\omega = 2\pi;$$

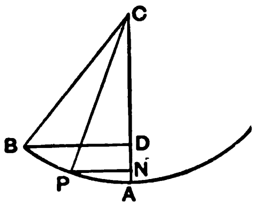
$$\therefore t = 2\pi\sqrt{\frac{h}{g}}.$$

If the circle be so small that h is sensibly equal to l, then

$$t=2\pi\sqrt{\frac{l}{g}}.$$

(9.) *A particle of mass* m *is suspended from* C *and oscillates under the action of its own weight, required the velocity at any point.*

Let B be the extreme position. When the mass has descended to P, the energy spent is mg.DN, and the whole is accumulated;



$$\therefore\ mg.\text{DN}=\tfrac{1}{2}mv^2.$$

Now $\text{DN}=\text{CN}-\text{CD}$

$$=(l\cos\theta-l\cos a);$$

$$\therefore\ v^2=2gl(\cos\theta-\cos a).$$

At the lowest point, therefore,

$$v^2=2gl(1-\cos a).$$

Regarding the potential energy at A as zero, the potential energy at P is mg.AN, and the kinetic is mg.DN; the sum of the two is therefore mg.DA, and is constant.

Remark.—Since $\cos\theta=1-2\sin^2\frac{\theta}{2}$ and $\cos a=1-2\sin^2\frac{a}{2}$;

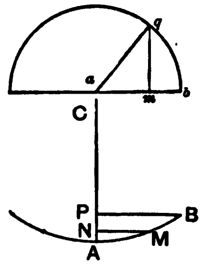
$$\therefore\ \cos\theta-\cos a=2\left(\sin^2\frac{a}{2}-\sin^2\frac{\theta}{2}\right).$$

Now if the arc AB be very small, we may write $\sin\frac{a}{2}=\frac{1}{2}\frac{\text{AB}}{l}$ and $\sin\frac{\theta}{2}=\frac{1}{2}\frac{\text{AP}}{l}$;

$$\therefore\ v^2=\frac{g}{l}(\text{AB}^2-\text{AP}^2).$$

(10.) *The Simple Pendulum.—A heavy particle is suspended from a fixed point C, and describes small oscillations under the action of its own weight; required the time of a complete oscillation.*

Let AB be the arc described, B being the initial position of the particle, A the lowest, and M any other position. Let BP and MN be perpendicular to CA.



Let *ab* be a horizontal straight line equal to the arc AB, and let *m* be a point in *ab*, which moves always with the same velocity as M moves along AB, so that $am = \text{AM}$.

With *a* as centre and *ab* as radius, describe a semicircle, draw *mq* perpendicular to *ab*, and join *aq*.

Now we will show that as *m* moves along *ab*, *q* moves with uniform velocity along the semicircle *bq*.

The velocity of the particle M, or the point *m*, is given by the equation

$$v^2 = 2g.\text{PN} = 2g\,(\text{AP} - \text{AN}).$$

But as the arc AB is supposed small, by a property of the circle $\text{AM}^2 = 2l\text{AN}$;

$$\therefore\ v^2 = \frac{g}{l}\,(\text{AB}^2 - \text{AM}^2)$$

$$= \frac{g}{l}\,(ab^2 - am^2)$$

$$= \frac{g}{l}\,(aq^2 - am^2)$$

$$= \frac{g}{l} qm^2\,;$$

$$\therefore\ v = \sqrt{\frac{g}{l}} \cdot qm.$$

Now in Δamq, *aq* is perpendicular to the actual velocity of *q*, *am* to its vertical component, and *qm* to its horizontal component. Also the horizontal component, which

is the same as the velocity of m, is proportional to mq, being $mq.\sqrt{\frac{g}{l}}$; hence by the triangle of velocities the velocity of q is $\sqrt{\frac{g}{l}}\cdot aq$, and is constant.

The angular velocity of q therefore is

$$\omega = \frac{v}{aq} = \sqrt{\frac{g}{l}}.$$

Hence if t be the time of a whole revolution of q, and consequently of a complete oscillation of M,

$$\omega t = 2\pi;$$

$$\therefore t = 2\pi\sqrt{\frac{l}{g}}.$$

EXERCISE XII.

1. A railway carriage weighing 5 tons passes round a curve, the radius of which is 250 yards; when the carriage is moving at the rate of 20 miles an hour, what is the outward pressure on the rails?

2. What ought to be the difference of level of the two rails to bring the whole pressure perpendicular to the plane of the rails, the distance between them being 57 inches?

3. A weight of 3 pounds is attached by a string, 3 feet long, to a point in a smooth horizontal table, and made to revolve round the point five times in three seconds; find the tension of the string?

4. Find the force towards the centre required to make a body weighing 2 pounds move uniformly in a circle whose radius is 5 feet, with such a velocity as to complete a revolution in 5 seconds.

5. A stone of 1 pound weight is attached to a string $4\frac{1}{2}$ feet long, having one end fixed, and whirled round horizontally; find the time of revolution when the tension of the string is 9 pounds.

6. A system, consisting of two bodies P and Q connected by a string, is in motion on a smooth horizontal table P and Q, describing circles with uniform velocities; determine the position of the point in the string which does not move.

7. A string 2 feet long can just support a weight of 16·1 pounds without breaking; one end of the string is fixed to a point on a smooth horizontal table; a weight of 4 pounds is fastened to the other end, and describes a circle with uniform velocity round the fixed point as centre; determine the greatest velocity which can be given to the weight so as not to break the string.

8. If a weight of 2 pounds and a thread three feet long form a simple pendulum which vibrates through an angle of 60°, find the greatest tension of the thread.

9. A piston is placed in the middle of a closed cylinder, there being equal quantities of air on each side; it is then moved halfway towards one end and let go. The pressure of the air on each side of the piston varies inversely as the volume; show how to find approximately the greatest velocity—

1st, when the cylinder is horizontal;
2nd, when it is vertical.

10. An elastic thread 5 feet long will just support a weight of 10 pounds when stretched to twice its length; if the tension varies as the stretched length, find the potential energy of the thread when stretched to thrice its length.

11. Two men put a railway waggon weighing 5 tons into motion by exerting on it a force of 80 pounds. The resistance of the waggon is 10 pounds per ton, or altogether 50 pounds; how far will the waggon have moved in 1 minute?

Calculate at what fraction of a horse-power the men are working at 60 seconds after starting.

12. A railway train is moving smoothly along a curve at the rate of 60 miles an hour, and in one of the carriages a pendulum, which would ordinarily oscillate seconds, is observed to oscillate 121 times in two minutes. Show that the radius of the curve is very nearly 2 furlongs.

Suppose a stone to be dropped from the window of this carriage, find approximately how far from the rail it will fall.

13. A piston, weighing 18 pounds and measuring a fourth of a square foot in area, fits tightly into a smooth vertical cylinder closed at the lower end, and is supported by the compressed air within at a height of 16 inches from the base; if the piston be raised 8 inches and then let fall, find approximately its greatest

velocity, the pressure of the air outside being 15 pounds to the square inch.

14. A pendulum is found to make 640 vibrations at the equator in the same time as it makes 641 at Greenwich; if a string hanging vertically can just sustain 80 pounds at Greenwich, how many pounds can the same string sustain at the equator?

15. A shot whose mass is m pounds issues from a gun, whose bore is smooth and a feet in length, with an initial velocity of v feet per second. Assuming the pressure of the gases which have impelled the shot to be constant, find the magnitude of the pressure, and determine the time which the shot has occupied in traversing the bore.

CHAPTER XI.

MACHINES.

119. Introduction.—An instrument for making a force which is applied at one point practically available at some other point is termed a machine; in other words, *a machine is an instrument for the transmission of energy.*

In the action of a machine the following three things take place:—

(i.) Some natural source of energy communicates motion and force to a piece of the machine, called the *prime mover*, at a point called the *driving point.*

(ii.) The motion and force are transmitted from the prime mover through the train of parts to the *working piece*, and during that transmission the motion and force are modified in amount and direction so as to be rendered suitable for the purpose to which they are to be applied.

(iii.) The working piece by its motion and force produces some useful effect.

Now the modification of motion depends entirely on the geometrical relations of the parts of the machine, and not on dynamical principles. Hence, the theory of machines naturally divides itself into two parts; the first part treating simply of the modification of motion, and the second part treating of the combined modification of motion and force. The first is usually called pure mechanism, the second applied mechanics.

The second part of the theory of machines is included in the theory of energy explained in the preceding chapter; thus, for all machines the following rules are true:—

(i.) If all resistances except that at the working point can be neglected for any time, the energy exerted = the work done at the working point + the work accumulated in the moving parts. (§ 114.)

(ii.) If friction has to be considered, the energy exerted = the work done at the working point + the work done in overcoming friction + the work accumulated in the moving parts. (§ 114.)

(iii.) If the machine has arrived at a state of uniform motion, then for any portion of time no change is made in the kinetic energy of the moving parts; and therefore the last term in the preceding equation is zero. (§ 115, i.)

(iv.) If the machine be at rest under the action of a force at the driving point technically termed the *power*, and a resistance at the working point technically termed the *weight*, then, neglecting friction, we may imagine the machine to move through its position of rest in any way and thus obtain the following equation:—

The power multiplied by the small displacement of the driving point in the direction of the power = the weight multiplied by the corresponding small displacement of the working point. (§ 115, viii.)

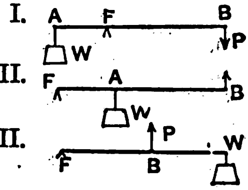
(v.) If the machine be in equilibrium in all positions, then it is not necessary to restrict the displacements to small quantities, for if we imagine the machine to be set in motion and left to itself, then between any two positions, the forces being always in equilibrium, there will be no gain or loss of kinetic energy, and therefore the work done by the power will equal that done on the weight (§ 115, viii.)

120. The Simple Machines.—The most simple machines are called *mechanical powers.* They are usually considered

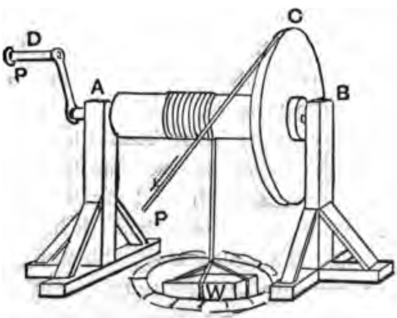
to be six in number; namely, the lever, the wheel and axle, the pulley, the inclined plane, the wedge, and the screw.

(i.) A *lever* is an inflexible rod moveable only about a fixed axis, which is called the *fulcrum*. The portions of the lever into which the fulcrum divides it are called the *arms* of the lever: when the arms are in the same straight line, it is called a *straight lever*, and in other cases a *bent lever*.

Levers are usually classified according to the position of the fixed point. If the fulcrum be between the power and the weight, the lever is of the *first* kind. If the weight be between the fulcrum and the power, the lever is of the *second* kind. If the power be between the fulcrum and the weight, the lever is of the *third* kind.



(ii.) The *wheel and axle* consists of two cylinders fixed together with their axes in the same line, the larger being called the wheel, and the smaller the axle. The cord by which the weight W is suspended is fastened to the axle AB, and then coiled round it, while the power P which supports the weight acts by a cord coiled round the circumference of the wheel C, by spokes acted on by the hand, as in the *capstan*, or by the hand acting on a handle D as in the *windlass*.



(iii.) The *pulley* consists of a small circular plate or wheel

which can turn round an axis passing through the centres of its faces, and having its ends supported by a framework, which is called the *block*.

A pulley, the axis of which is fixed in space, is termed a *fixed* pulley, and in mechanics serves the purpose only of changing the direction of the power. If the axis be moveable, the pulley is termed a *moveable* pulley.

(iv.) An *inclined plane* in mechanics is a rigid plane inclined to the horizon.

(v.) The *wedge* is a solid triangular prism made of hard material such as iron or steel, and is used for separating two bodies, or two parts of the same body, which adhere powerfully to each other. The edge of the wedge is introduced between the parts of the substance, and it is then driven forward by smart blows of a hammer applied at its back.

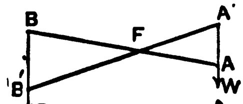
(vi.) The *screw*, in its simplest construction, consists of a cylinder with a uniform projecting thread traced round its surface, and making a constant angle with lines parallel to the axis of the cylinder. This cylinder fits into a block pierced with an equal cylindrical aperture, on the inner surface of which is cut a groove the exact counterpart of the projecting thread.

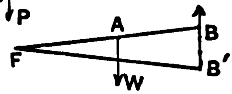
It is easily seen from this description, that when the cylinder is introduced into the block, the only manner in which it can move is backwards or forwards by revolving about its axis, the thread sliding in the groove.

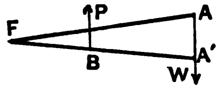
121. Equations of Equilibrium of the Simple Machines.— In investigating the equations of equilibrium of the simple machines, the imaginary displacement will, for convenience, be so taken that the forces which are in equilibrium in one position will also be so in any other position passed through in the displacement, so that we can apply (iv.) and (v.) of section 119, and need not necessarily choose a small displacement.

(i.) *The Lever.*—Let the forces be called, respectively, P

and W, and let them be applied at points A and B. Take two positions of the lever AB, A′B′, such that AA′ is a vertical line; then BB′ will also be a vertical line;







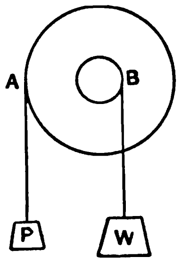
$$\text{and } P.BB' = W.AA'.$$

$$\text{But } AA' : BB' :: AF : BF;$$

$$\therefore P.BF = W.AF.$$

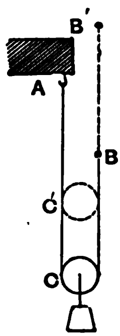
In other words, the condition of equilibrium in the straight lever is, that the two forces must be inversely proportional to the distances of their points of application from the fulcrum.

(ii.) *The Wheel and Axle.*—Let C be the circumference of the wheel, and c that of the axle. If the wheel is turned round, the forces which are in equilibrium in one position are evidently also in equilibrium in all positions. Suppose the wheel and axle to turn once round, then P will have descended through a distance equal to C, while W will have ascended a height equal to c.



$$\therefore P.C = W.c,$$

or since the circumferences C, c are proportional to the radii R, r;



$$\therefore P.R = W.r.$$

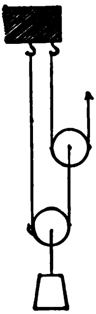
(iii.) *The Pulley.*

(1.) Suppose we have a single pulley about which the flexible and inextensible cord ACB is wrapped, and suppose its free portions to be parallel. Suppose the weight to rise through a height h, then the point B will rise through a height h in consequence of the rise of the pulley, and also through a height h in consequence of the shortening

of the string on the left. Therefore the point of application of the power will move through a space $2h$; hence,

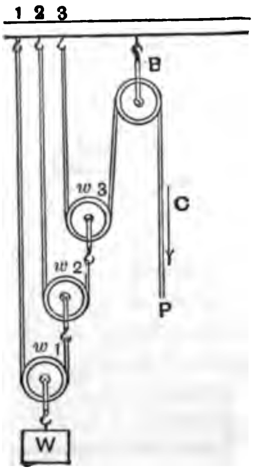
$$\text{W}.h = \text{P}.2h,$$
$$\text{or } \text{W} = 2\text{P}.$$

Also, when B moves through any space in the direction of the cord, the pulley moves in the same direction, through half the space.



(2.) If there are two cords and two pulleys, the ends AA′ being fixed, and the other end of AC being attached to the pulley C′; then, if all free parts of the cord are parallel, when C is moved through a height h, C′ moves in the same direction through twice this space, and carries with it one end of the cord AC. But B moves through twice the space C′ moved through, that is, four times the space h;

$$\therefore \text{W}.h = \text{P}.4h,$$
$$\text{or } \text{W} = 4\text{P}.$$



(3.) Similarly we may extend this to any number of pulleys, if they are arranged in the above manner. Similar considerations enable us to determine the relative motions of all parts of other systems of pulleys and cords as long as all the free parts of the cords are parallel.

Of course, if a pulley is *fixed*, the motion of a point of one end of the cord *to* or *from* it, involves an equal motion of the other end *from* or *to* it.

Let there be n moveable pulleys arranged with separate strings, one end of each string being attached to the beam at the points 1, 2, 3, etc.,

and let it be required to find the equation of equilibrium, taking into account the weights w_1, w_2 . . . w_n of the pulleys.

Imagine W to rise 1 foot, then the work done $=$ W :

then w_1 will rise 1 foot, and work done $= w_1$

w_2 „ „ 2 feet, „ „ „ $= 2w_2$

w_3 „ „ 4 feet, „ „ „ $= 4w_3$

. . . .

w_n „ 2^{n-1} feet, „ „ „ $= 2^{n-1}w_n$

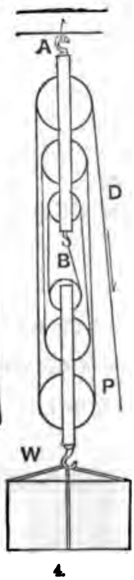
P will descend 2^n feet, „ „ „ by it $= 2^n$P ;

$$\therefore 2^n\text{P} = \text{W} + w_1 + 2w_2 + 2^2w_3 + \ldots + 2^{n-1}w_n.$$

(4.) If there are n pulleys in one block, as in the figures, when W rises 1 foot, each of the $2n$ parts of the string is shortened 1 foot, and therefore P must descend $2n$ feet;

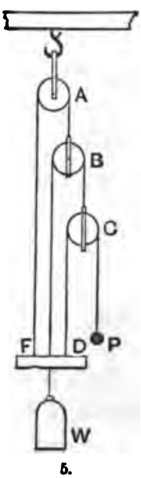
$$\therefore \text{W} = 2n\text{P}.$$

4.



4.

(5.) If there are different strings, each attached to the weight FD, as in the figure, when W rises 1 foot B descends 1 foot, C descends 2 feet in consequence of the fall of B, and 1 foot in consequence of the rise of W, or 3 feet in all. D descends 6 feet in consequence of the fall of C, and 1 foot through the rise of W, or 7 feet in all, and so on. But as the pulleys descend, they do work in the same direction



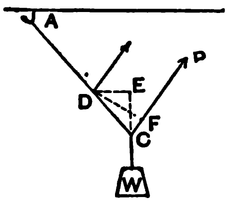
5.

as that of P; hence, we have, if there be n strings and therefore $n-1$ moveable pulleys, the lowest being the first,

$$\begin{aligned} W = & w_1(1+2+2^2+ \ldots +2^{n-2}) \\ & + w_2(1+2+2^2+ \ldots +2^{n-3}) \\ & \ldots\ldots\ldots \\ & + w_{n-1} \\ & + P(1+2+2^2+ \ldots +2^{n-1}); \end{aligned}$$

$$\therefore W = w_1 2^{n-1} + w_2 2^{n-2} \ldots + w_{n-1} 2 - (w_1 + w_2 + \ldots + w_{n-1}) \\ + P(2^n - 1).$$

(6.) Let there be one moveable pulley at C, and let the two parts of the thread make an angle ACB, the magnitude of which is 2α. Let P be the tension of the string. The pulley at C is acted on by three forces, P along CA, P along CB, and W vertical.



Imagine W to move to D in the line CA and CB to move parallel to itself.

Draw CE vertical, DE horizontal, and DF perpendicular to CF.

Then the work of P along $AC = P.CD$,

$$\text{work of P along } CP = P.CF = P.CD. \cos 2\alpha$$

$$\text{work of W vertically} = W.CE = W.CD. \cos\alpha;$$

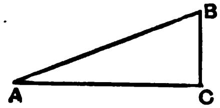
$$\therefore P.CD + P.CD. \cos 2\alpha = W.CD. \cos\alpha;$$

$$\therefore 2P. \cos\alpha = W.$$

This is the same equation as would result from the application of the parallelogram of forces.

(iv.) *The Inclined Plane.*—First, suppose the force to act in a direction parallel to the plane. Imagine the

body to be moved from A to B, then the weight will have been lifted through a height BC and the power P will have acted in the line of its direction through a space AB. Therefore, by equating the works,

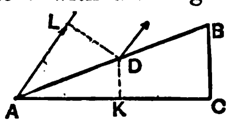


$$W.BC = P.AB.$$

Secondly, let us suppose the force to be horizontal. Imagine, as before, that the body moves from A to B, then the weight will have been lifted through a height BC, and the power will have acted in the line of its direction, that is to say, horizontally, through a space equal to AC. Hence, by equating the works, we have

$$W.BC = P.AC.$$

Finally, let the force make an angle θ with the length of the plane, the inclination of the plane being α, and let R be the reaction of the plane.



Imagine the body to move from A to D, the force remaining parallel to itself.

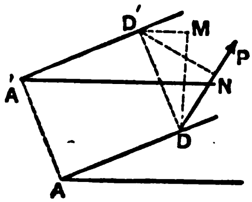
Draw DK vertical to the base AC, and DL perpendicular to P.

$$\text{Work by } P = P.AL = P.AD\cos\theta.$$
$$\text{Work by } W = W.DK = W.AD\sin\alpha.$$
$$\text{Work of } R = 0,$$
$$\therefore P\cos\theta = W\sin\alpha.$$

If friction has to be taken into account, and it is equal to a fraction m of the resistance R, then we must add the work of friction, namely, $\pm m R.AD$;

$$\therefore P\cos\theta \pm mR = W\sin\alpha.$$

To obtain an equation involving R, imagine the body and the plane to move in a direction perpendicular to the plane, the directions of the force and plane remaining unaltered during the motion. Draw D′N perpendicular to the direction of the force DM vertical, and D′M horizontal, then $\angle MDD' = a$, $\angle D'DN = 90° - \theta$.

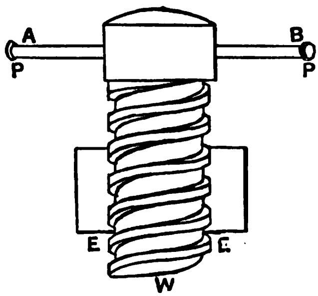


$$\therefore \text{R.DD}' + \text{P.DN} = \text{W.DM},$$

$$\therefore \text{R.DD}' + \text{P.DD}' \sin\theta = \text{W.DD}' \cos a.$$

$$\therefore \text{R} + \text{P} \sin\theta = \text{W.} \cos a.$$

(v.) *The Screw.*—Suppose W to be the weight acting on the cylinder (including the weight of the cylinder itself), and P to be the power acting at the end of an arm AB at right angles to the axis of the cylinder. (The block E is supposed to be firmly fixed, and the axis of the cylinder to be vertical.)



Suppose the power P to act at one end of the lever only at a point which describes a circle of circumference C, and radius l, about the axis of the screw. Let d be the distance between two consecutive threads. Imagine the screw to turn round once, then, the power acting always at right angles to the lever, the work done by the power is P.C, and that done on the weight is W.d, hence

$$\text{P.C} = \text{W.}d,$$

or the power is to the weight as the distance between the threads is to the circumference of the circle described by the power.

122. Mechanical Advantage.—By the *mechanical advantage* of any machine is meant the ratio of the weight to the power, when in equilibrium; thus if a power of 5 pounds sustains a weight of 80 pounds, the mechanical advantage is $80 \div 5$, or 16. Thus the mechanical advantage of a simple machine may be found from the preceding equations of equilibrium.

To find the mechanical advantage of any combination of machines.

In a combination of machines, the weight of the first machine is the power of the second, the weight of the second the power of the third, and so on. Let a, b, c, be the separate advantages of three machines in combination. Let P and Q be the power and weight in the first, Q and R those in the second, R and W those in the third; then, $Q = aP$, and $R = bQ$, and $W = cR$; hence, multiplying together $QRW = abcPQR$; $\therefore W = abcP$. Therefore abc represents the mechanical advantage of the combination; that is, the advantage of the combination is equal to the product of the separate advantages of the component machines.

In like manner it may be shown that the mechanical advantage of the combination of any number of machines is equal to the product of the separate advantages of the component machines.

123. Differential Machines.—The ratio of the velocity of the driving point to the velocity of the working point may be increased by causing the motion of the working point to depend on two motions in opposite directions very nearly equal to one another. A machine possessing this feature

is termed a *differential* machine. The practical advantages of such a machine will be seen from the following descriptions :—

(i.) *Differential Wheel and Axle.*—In the simple wheel and axle, whatever be its form or peculiarity, its *mechanical advantage* depends on the ratio of the radius of the wheel to the radius of the axle. There are then two, and only two ways, in which this advantage may be increased, viz., by increasing the leverage of the power, or by diminishing the radius of the cylinder which supports the weight. Now, although, in a theoretical view, we can conceive the leverage of the power increased without limit, or the thickness of the cylinder diminished without limit, yet there is a practical limit to these changes.

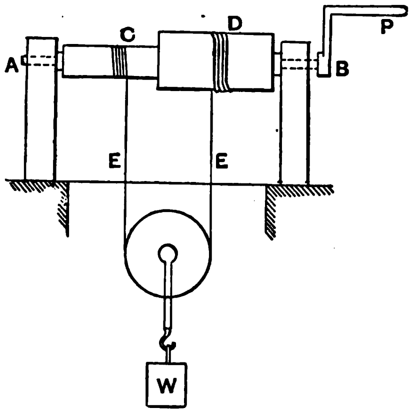
If we give to the power a considerable leverage, the machine will become unwieldy, a practical disadvantage which will more than counterbalance anything which can be gained by the increased power.

If, on the other hand, we attempt to increase the advantage by diminishing the thickness of the axle, we diminish the strength of that part of the machine which must support the weight.

In cases, therefore, where great resistances are to be overcome, it is a problem of considerable importance to find a method, by which, without rendering the machine more complex or unwieldy, sufficient strength may be preserved and a high degree of power may be gained.

All these ends are attained by a simple modification of the wheel and axle. The axle, or cylinder AB, consists of two parts, the diameter of one part being less than that of the other. A moveable pulley is attached to the weight, and round the wheel of the pulley the rope which raises the weight is passed, and is coiled in the *same direction* on the thicker and thinner parts of the axle. When the axle

is turned in such a direction that the rope shall be coiled upon the thicker part, it will necessarily be rolled off the thinner part. Upon every revolution of the wheel, therefore, a portion of the rope equal to the circumference of the thicker part will be *drawn up*; but, at the same time, a portion, equal to the circumference of the thinner part, will be *let down*. On the whole, therefore, the effect of one revolution will be to shorten the entire length of that part of the rope, by which the weight is suspended, by a length equal to the difference between the circumferences of the thicker and thinner parts of the axle. Hence, every revolution of the cylinder raises the weight through a space equal to half the difference between the circumferences of the two axles.



Let c be the circumference described by the power, a the circumference of the larger axle, and b that of the smaller. Let P be the power, and W the weight, and when there is equilibrium, imagine the wheel to turn round once and equate the works;

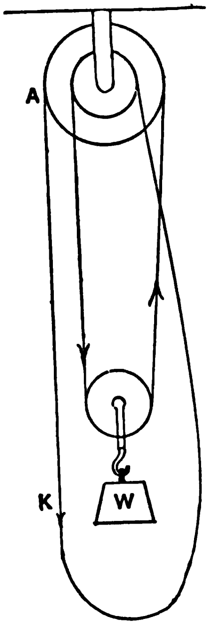
$$\therefore\ c.\mathrm{P} = \frac{a-b}{2} \cdot \mathrm{W},$$

and the mechanical advantage $= \dfrac{\mathrm{W}}{\mathrm{P}} = \dfrac{2c}{a-b}$.

This may evidently be increased to any extent by making the axles A and B as nearly equal as we please without un-

duly reducing the strength of the axle. If the wheel and larger axle be of the same size, the equation becomes $a.P = \frac{1}{2}(a-b)$ W.

(ii.) *The Differential Pulley.*—A modification of the differential wheel and axle has been termed the differential pulley. If the two axles be made so short that there is but breadth enough for a groove to receive a chain once round each, and if the two ends of the chain, instead of being attached to the axles, are joined together, the result will be the differential pulley. The power is applied to that side of the loop which comes from the larger axle (the part AK in the figure) so that the larger axle is also the *wheel.* The wheels are furnished with projections, which by fitting into the links of the chain prevent it from slipping.

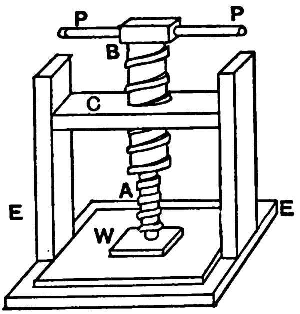


If the pulley goes round once, the power P acts through a distance a, and the weight through a distance $\frac{1}{2}(a-b)$; hence, when there is equilibrium, $a.P = \frac{1}{2}(a-b)$ W.

(iii.) *The Differential Screw.*—The proportion of the weight to the power, or the mechanical advantage of the screw, depends on the proportion of the circumference described by the power, to the distance between two contiguous threads. Hence, it is evident that the efficacy of the screw may be increased, either by increasing the length of the lever by which the power acts, or by diminishing the distance between the threads. To both of these there are, however, practical limits, similar to those mentioned in the case of the wheel and axle.

By a contrivance, termed Hunter's screw, all the requisite strength and compactness in the machine are secured, together with an almost unlimited degree of mechanical advantage.

This contrivance is represented in the figure; EE is a strong frame in which a board W moves, so that when it is forced towards the bottom, it will exert a pressure on any substance placed between it and the bottom. To this moveable board is attached a cylinder A, on which the thread of a screw is raised. C is a fixed nut through which a screw B plays. The screw B is a hollow cylinder, the interior of which is a nut adapted to receive the screw A.



When the screw B is turned once round, it advances through the nut C through a space equal to the distance between two contiguous threads; so that, if the screw A were not supposed to act, the board W would advance towards the bottom, through a distance equal to the distance between two contiguous threads of the screw B. But while the screw B advances through the nut C by its revolution, the very same cause makes the screw A move towards C through a space equal to the distance between two contiguous threads of A; or, by turning B, the nut contained in the inner concave surface of B is turned upon the screw A. Now, if the threads of the two screws A and B were perfectly equal, the effect of these two motions would be, that the board W would retain its position.

But if we suppose the interval between the threads of the screw A to be somewhat less than the interval between the

threads of the screw B, one revolution will move the screw B *downwards*, through a space equal to the interval between its threads, while the screw A will be moved within the screw B *upwards* through a space equal to the interval between *its* threads. The combined effect will be, that the screw A, and the board W to which it is attached, will be moved *downwards* through a space equal to the difference of the distances between the threads of the two screws.

Thus, if c be the radius of the circle described by P, and b, a the distances between two threads in the screws B, A, respectively; then whilst P makes one revolution W will descend through b, in consequence of the screw B descending through b, but it will also rise through a in consequence of the screw A making one turn within B; *i.e.*, it will descend through $b-a$.

$$\therefore \mathrm{P}.2\pi c=\mathrm{W}.\ (b-a);$$

$$\therefore \frac{\mathrm{W}}{\mathrm{P}}=\frac{2\pi c}{b-a}.$$

Hence, by making b as nearly equal to a as we please, the *mechanical advantage* may be increased to any extent without unduly weakening the threads of the *screws*.

Example 58.—*A power* P *draws up a weight* Q *by means of a wheel and axle, the radii being respectively* p *and* q. *Find the time,* t, *in which* Q *will ascend* s *feet, the velocity of* Q *at the end of this time, and the acceleration during the motion, neglecting the inertia of the wheel and axle.*

While Q ascends s feet and acquires a velocity v, P descends $\frac{ps}{q}$ feet and acquires a velocity $\frac{pv}{q}$.

By equating the work expended to the work effected and accumulated

$$\frac{\mathrm{P}ps}{q}=\mathrm{Q}s+\frac{1}{2}\frac{\mathrm{P}}{g}\left(\frac{pv}{q}\right.\ +\frac{1}{2}\frac{\mathrm{Q}}{g}v^2,$$

$$\therefore v^2 = \frac{2gqs(Pp - Qq)}{Pp^2 + Qq^2}.$$

But if f be the acceleration of Q

$$\therefore v^2 = 2fs \quad . \quad . \quad . \quad (\S\ 14\ B).$$

$$\therefore f = \frac{gq(Pp - Qq)}{Pp^2 + Qq^2},$$

$$\text{and } t = \frac{v}{f} = \sqrt{\frac{2s(Pp^2 + Qq^2)}{gq(Pp - Qq)}}.$$

EXERCISE XV.

1. A straight lever, whose weight is 5 pounds and length 6 feet, rests on a fulcrum when a weight of 1 pound is suspended from one extremity; find the position of the fulcrum and the pressure on it.

2. Two weights of 4 pounds and 7 pounds, respectively, balance on a uniform heavy lever, the arms of which are in the ratio of 3 to 2; find the weight of the lever.

3. If the same body be weighed successively at the two ends of a false balance whose arms are of unequal length, its true weight is the square root of the product of the apparent weights.

4. Find the true weight of a substance which, when placed in one scale of a balance, seems to weigh 140 grammes, and when in the other appears to weigh 154·35 grammes.

5. If in a balance one arm be ·98 of the other, and a body placed in the scale of the shorter arm balance 14·7 kilogrammes in the other scale, find the true weight of the body.

6. A uniform rod 4 feet long is balanced about its middle point, and weights of 3 pounds and 5 pounds, respectively, are suspended from the extremities; find the acceleration with which the weights will begin to move, neglecting the inertia of the rod.

7. The arms of a lever are 12 inches and 18 inches, respectively, and the weights are 12 pounds and 9 pounds; find the accelerations with which they begin to move.

8. A capstan turned by two horses is used to draw in a boat; the levers to which the horses are attached are 12 feet long, and the

radius of the axle is 18 inches. When each horse is pulling with a force of $7\frac{1}{2}$ cwts., find the tension of the cord attached to the boat.

9. A uniform bent lever, the weights of whose arms are 5 pounds and 10 pounds, rests with the shorter arm horizontal; what weight must be attached to the end of the short arm that the lever may rest with the long arm horizontal?

10. If a weight of 10 pounds raise a weight of 50 pounds by means of a wheel and axle, the circumferences being as 6 to 1, find the velocities of the weights when the larger has ascended 10 feet.

11. The radius of the wheel being 20 inches, and that of the axle 4 inches, how long will it take 101 ounces to raise 500 ounces through 20 feet; and what will then be the velocity of the 500 ounces?

12. A weight is supported on an inclined plane by two forces, each equal to half the weight, one acting along the plane, and the other horizontally; show that the inclination of the plane is 45°.

13. The length of the lever of a screw is 21 inches, and the distance between the threads is $\frac{1}{4}$ of an inch; what power applied at the circumference of the screw will support a weight of 110 pounds?

14. In a common press the diameter of the screw is 6 inches, the distance between the threads $\frac{3}{8}$ of an inch, and the length of the lever, measured from the axis, is 4 feet; what power will support a resistance of 352 pounds?

15. If the circumference described by the end of the lever be 10 feet, the power 10 pounds, and there be three threads in 2 inches, find the resistance supported.

16. In the system of pulleys where each string is attached to the weight, if one of the strings be nailed to the block through which it passes, show that the power may be increased up to a certain limit without producing motion. If there be three pulleys, and the action of the middle one be checked in the manner described, find the tension of each string for given values of P and W.

17. With two moveable pulleys, each weighing 1 ounce, and having separate strings attached by one end to the beam, how long will it take 8 ounces to raise 27 ounces through 12 feet; and what will then be the velocities of the pulleys, all parts of the strings being parallel, and friction being neglected?

18. By means of a moveable pulley-block with three wheels, a weight of 1 cwt. is let down 10 feet, the power being 18 pounds; find the final velocity and time of descent.

19. Find the pressure on the beam during the descent.

20. If, with a differential wheel and axle, when the radii of wheel and axles are respectively 24 and 4·3 and 4·2 inches, the weight is 200 pounds and the power $\frac{1}{2}$ pound; find the velocities acquired—

(1.) When the weight has ascended 10 feet.

(2.) When the weight has ascended for 10 seconds.

21. A weight of 100 pounds placed on an inclined plane, rising 72 in 1297, is acted on by a constant horizontal force of $5\frac{1}{2}$ pounds; find the time the weight will take to descend 216 feet 2 inches.

22. A man, sitting upon a board suspended from a single moveable pulley, pulls downwards at one end of a rope which passes under the moveable pulley and over a pulley fixed to a beam overhead, the other end of the rope being fixed to the same beam; what is the smallest proportion of his whole weight with which the man must pull in order to raise himself?

23. With what force would he require to pull upwards if the rope, before coming to his hand, passed under a pulley fixed to the ground, as well as round the other two pulleys?

CHAPTER XII.

FRICTION.

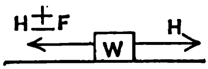
124. Introduction.—When a rigid system of material particles, acted on by a number of forces, is in contact with another system, forming for the first a *supporting surface*, the reactions at the points of contact are normal to the surfaces at those points (§ 64, i.). If the surfaces were perfectly smooth, therefore, these reactions would be perpendicular to the general contour; but, in reality, all surfaces are more or less rough, that is to say, have certain small prominences and hollows, so that normals at the points of contact are not necessarily normal to the general curve of the surface. The reactions, therefore, have not only components normal to the general contour, but also components tangential to it. The sum of these latter components is termed the *friction* between the surfaces.

Friction is therefore a pure resistance, incapable of moving a machine from a position of rest, or of increasing its velocity when in motion, but which, whenever two surfaces in contact move with different velocities, tends to diminish the velocity of the more rapid. It is not a simple force, but is the resultant of actions which take place between the molecules of the surfaces in contact. We know very little of the nature of these actions, except that their resultant is a force which obeys the general laws of natural forces. We shall content ourselves at present with the consideration of

the methods of measuring this force, postponing the full discussion of the peculiarities of friction until we come to consider the Dynamical Theory of Heat.

When friction is in operation, a less force is required to produce the equilibrium of a body acted on by other forces than when it is absent; and friction preserves equilibrium, when without it, motion must ensue.

If, for example, a body lying on a rough horizontal surface be acted on by a horizontal force H towards the right, the momentum *mf.* produced per second will be less than that which would result from the force H acting alone; in other words, it will be such as would result from a force H−F, or



$$m.f = \text{H} - \text{F}.$$

Here F is the resistance due to friction. A force equal to H−F applied in the opposite direction, that is to say, towards the left, will produce equilibrium. If, therefore, the body be at rest, a force H−F towards the left will keep it at rest, and it will be then just on the point of moving towards the right. But H+F will also keep it at rest, the body being then just on the point of moving towards the left. And any force between H+F and H−F will keep it at rest, but in this case the friction called into play is the excess or defect of this force over H, and, of course, is not the maximum. If the force applied towards the left be H, that is to say, be exactly equal and opposite to the resultant of the other forces acting on the body towards the right, then the friction is zero.

The frictional forces, therefore, are produced only when the other forces, acting on the system, contain components in the plane of contact, and the resultant of these components, when the system is at rest, is equal and opposite to the resultant force of friction. Hence, the resistance of

friction may be variable in magnitude and direction, can only lie in the plane of contact, and cannot exceed a certain maximum, which is constant so long as the materials are the same. This maximum value is taken as the measure of friction.

The motion of one rigid surface on another may be a *sliding* or *rolling* motion, and the sliding may be *progressive*, as when the same points of the moving system come into contact with consecutive points of the supporting surface, like the motion of a sledge; or *rotatory*, when consecutive points of the moving system come into contact with the same points of the supporting surface, like the motion of an axle in its bearings.

In *rolling* motion consecutive points of the moving system come into contact with consecutive points of the supporting surface, the contact between the surfaces taking place along a straight line, or at a point. The nature of the friction varies with the kind of relative motion of the surfaces.

125. Sliding Friction.—The laws of sliding friction have been determined by experiment, and are as follows:—

1. The friction increases with the roughness of the surfaces in contact.

2. The resistance of friction is proportional to the mutual pressures between the surfaces in contact.

3. The resistance of friction is independent of the extent of the surfaces in contact when they contain neither points nor edges.

4. When the body is set in motion from rest, the resistance of friction is greater than when simply an existing motion is maintained. In the latter case, however, the resistance of friction can be more easily determined; and as the transition from rest to motion can be effected by a very slight impulse, friction is usually calculated as if the body were already in motion.

5. The resistance of friction is independent of the velocity when the system is in motion.

These laws hold good so long as the bodies during the experiment become neither heated nor suffer any perceptible change of form.

126. Illustration.—Let CB be a plane moveable about a hinge at C, and resting on an arc AB, by means of which its inclination can be altered. Let a box, into which different weights can be placed, rest on the plane, and let a cord attached to the box pass over a pulley at B.

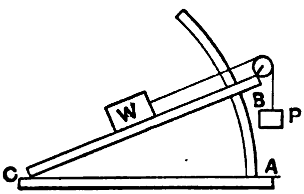
First fix the plane horizontally, and when W_1 is the weight of the box and its contents, let F_1 be the force, which will make the box on the point of sliding. Then increase the weight to W_2, and let F_1 then become F_2, that is to say, let F_2 be the force which, when aided by the slightest impulse, will cause the box to move; then it will be found that

$$\frac{F_1}{W_1}=\frac{F_2}{W_2}.$$

This fraction is therefore constant; let it be called μ, and since the weight acts normally to the plane, let N be written for W;

$$\therefore F=N.\mu.$$

Now let the cord be removed, and let the plane be inclined until the body is on the point of sliding. Then if ϕ be the angle, F the friction, and N the normal pressure on the plane, by § 121 iv., we have



$$F=\sin\phi,\ N=W\cos\phi;$$

$$\therefore F=N\tan\phi,$$

$$\text{or, } F=\mu.N$$

$$\text{where } \mu=\tan\phi.$$

The angle ϕ, which is the greatest angle at which the

plane will support the body, is termed the *limiting angle of repose*; hence, the last equation states that the co-efficient of friction is equal to the tangent of the limiting angle of repose.

If the body move through a space s in the direction of friction, the work done against friction is evidently $s.\mu N$, where N is the normal pressure between the surfaces.

127. Revolving Axles.—The friction between axles and their bearings is a force tangential to the circumference of the axle opposing the revolution. Axles may be vertical, inclined, or horizontal. In the first and second kinds the surfaces in contact must be either planes or surfaces of revolution having the same axis as the axle; and in the third kind they must be cylindrical.

Since friction opposes the revolution of the axle, the quantity which has to be determined here is the moment of the friction about the axis of revolution. If the moment be M, then for one revolution, by § 85, if W be the work done, we have

$$W = M . 2\pi.$$

To calculate the moment M, we must divide the surface into elementary areas, find the normal pressure on each, multiply by the distance of the area from the axis, and, finally, sum the products thus obtained. If a be an element of area, p the normal pressure at the element per unit area, y the distance from the axis, then

$$M = \mu \Sigma(apy) \quad . \quad . \quad . \quad . \quad (1).$$

If P be the resultant pressure on the axle, and θ the angle between the normal at an element a and the direction of P, then evidently P must be equal to the sum of the components in the direction of P of the normal pressure on every element, or

$$P = \Sigma(a.p \cos\theta) \; . \quad . \quad . \quad . \quad (2).$$

For new axles the normal pressure p is generally assumed to be constant for the surfaces in contact.

The work of the friction at any point while the axle revolves through an angle θ is, by § 85, the moment of friction multiplied by θ;

$$\therefore \text{work} = \mu.a.p.y.\theta.$$

128. Rolling Friction.—There remains to be considered the resistance which friction opposes to the further motion of a *rolling* body.

The laws found by experiment with regard to this resistance are :—

(1.) The resistance of friction in rolling motion is always smaller than in sliding motion.

(2.) The resistance is directly proportional to the normal pressure, and inversely to the radius of the rolling body.

Two similar cylindrical bodies weighing P_1 and P_2, and with radii r_1 and r_2, roll upon the same base AB, to find the relation between the forces of friction.

From the above laws we get, when F_1 and F_2 denote the resistances of friction in bodies with the same radius that

$$F_1 : F_2 = P_1 : P_2 ;$$

and when they denote the resistances of friction in bodies of the same weight but of unequal radii, we have

$$F_1 : F_2 = r_2 : r_1 ;$$

$$\therefore F_1 : F_2 = P_1 r_2 : P_2 r_1 ;$$

$$\therefore \frac{F_1 r_1}{P_1} = \frac{F_2 r_2}{P_2}.$$

In bodies of the same material this quotient has therefore a constant value, named the *co-efficient of rolling friction*, which we shall denote by v.

Generally, therefore, $\frac{Fr}{P}=v$ and, for the moment Fr of the rolling friction, we have

$$Fr=vP,$$

when F denotes the resistance of friction, P the normal pressure producing friction, and r the radius of the rolling body. We assume here that the force overcoming friction has its point of application at the centre of gravity of the body, and acts in a direction parallel to the base.

EXERCISE XVI.

1. A body is supported on a rough plane by a force acting parallel to the plane; show that the force will be greatest when the inclination of the plane is the complement of the limiting angle of repose.

2. A body is drawn along a rough horizontal plane by means of a cord; what must be the inclination of the cord that the force required may be the least possible?

3. A hemisphere is supported by friction with its curved surface in contact with a horizontal and a vertical plane; find the limiting position of equilibrium. (The C.G. is $\frac{3}{8}$ of the radius from the centre.)

4. Enunciate the laws of friction and define the angle of friction.

A uniform ladder 10 feet long rests with one end against a smooth vertical wall and the other end on the ground, the co-efficient of friction being ·5; find how high a man whose weight is four times that of the ladder may ascend before the ladder begins to slip, the foot of the ladder being 6 feet from the wall.

5. A weight W of 12 pounds on a rough table is attached to a thread which passes over the edge of the table, and sustains a weight of 3 pounds; when the latter has descended through 5 feet the thread breaks, and W moves through 4 feet more and comes to rest; what is the co-efficient of friction?

6. A weight of 100 pounds is sustained on a rough plane, inclined at an angle α by a force F inclined at an angle β to the plane; if the greatest angle at which the body would rest is 45°, find the limits between which F must lie.

7. A ladder rests against a vertical wall, to which it is inclined at an angle of 45°; the centre of gravity of the ladder is at $\frac{1}{3}$ the length from the foot. The co-efficient of friction for the ladder and plane is $\frac{1}{3}$, and for ladder and wall $\frac{1}{4}$. If a man whose weight is half the weight of the ladder ascend it, find to what height he will go before the ladder begins to slide.

8. If the roughness of a plane which is inclined to the horizon at a known angle be such that a body will just rest supported on it, find the least force requisite to draw the body up.

9. Two rough bodies rest on an inclined plane, and are connected by a string which is parallel to the plane; if the co-efficient of friction be not the same for both, find the greatest inclination of the plane which is consistent with equilibrium.

10. If a locomotive working at 220·8 horse-power pulls a train, the whole weight of which is $41\frac{2}{3}$ tons, up an incline rising 76 in 1445 with a speed which increases regularly for a minute, and is then $22\frac{1}{2}$ miles an hour, find the co-efficient of friction.

11. A solid vertical axle of radius r revolves on its circular end. If P be the pressure along the axle, and M the moment of friction, show that

$$M = \tfrac{2}{3}\mu r P.$$

12. A solid horizontal cylinder of radius r revolves in a hollow cylinder which exactly fits it; show that

$$M = \tfrac{1}{2}\pi\mu r P.$$

CHAPTER XV.

MOMENTS OF INERTIA.

129. Introduction.—In the preceding chapters we have restricted our investigations to cases in which all the particles of each body have the same velocity; we are now about to consider some cases in which the velocities are not the same, as, for instance, in the motion of a rigid body rotating about an axis.

It has been shown that when a mass moves so that all the particles have the same velocity, the kinetic energy of the mass is measured by the product of half the mass by the square of the velocity, or $\frac{1}{2}mv^2$; now, it will be shown that when a rigid body rotates about an axis, the kinetic energy of the body at any instant may be represented by an expression of the same form, namely, half the product of the square of the angular velocity, by a quantity which is termed the Moment of Inertia of the mass about the given axis, or $\frac{1}{2}N\omega^2$. We shall in this chapter have to consider some of the properties of this quantity N, and to find its value for a few simple figures.

130. Definitions.—*Definition of Moment of Inertia.*—The sum of the products of the mass of each particle of a system by the square of its distance from any straight line is called the *Moment of Inertia* of the system about the line.

Definition of Radius of Gyration.—Let K be such a quantity that the moment of inertia is the whole mass multiplied by K^2, then K is called the Radius of Gyration. If N be the moment of inertia and M the mass, then $N = MK^2$.

Hence K is the distance from the axis of that point at which, if the whole mass were collected there, the moment of inertia would be unaltered.

We will use k for the radius of gyration when the axis is supposed to pass through the C.G., and K when it does not.

131. **Proposition XXIX.**—*To find the Kinetic Energy of a system of particles revolving with the same angular velocity about a given point.*

Let M_1, M_2, M_3, etc., be the masses of heavy particles rotating with the same angular velocity ω about a centre O. Let their distances from O be respectively r_1, r_2, r_3, and let the velocities be v_1, v_2, v_3,

$$\text{then } v_1 = r_1\omega,$$
$$v_2 = r_2\omega,$$
$$v_3 = r_3\omega.$$

The kinetic energy of the system

$$= \tfrac{1}{2}Mv^2 + \tfrac{1}{2}M_2v_2^2 + \tfrac{1}{2}M_3v_3^2 + \text{etc.} = \tfrac{1}{2}\omega^2(M_1r_1^2 + M_2r_2^2 + M_3r_3^2 + \text{etc.})$$

This expression may be writen thus:—The kinetic energy $= \frac{1}{2}\omega^2 N$. From the preceding definition, N is the moment of inertia. Hence the kinetic energy is half the product of the Moment of Inertia of the whole mass about the given axis, by the square of the angular velocity.

132. **Proposition XXX.**—*The moment of inertia of a system of heavy particles about a given axis is equal to the moment of inertia about a parallel axis through the centre of gravity, together with the product of the whole mass by the square of the distance of the centre of gravity from the given axis.*

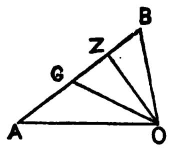
Let N be the moment of inertia about the given axis, and $\bar{N}$ that about an axis through the centre of gravity parallel to the first, let h be the distance of the centre of gravity from the first axis, then, M being the whole mass,

$$[N = \bar{N} + Mh^2].$$

Take a plane perpendicular to the axis at a point O, and project the positions of the particles upon it.

First let there be two heavy particles, and suppose P and Q to be their masses, and A and B to be the projections of their respective positions. Let G be the projection of their centre of gravity. Let OZ be the perpendicular from O on AB.

Then, by a well-known geometrical proposition,



$$OA^2 = AG^2 + OG^2 + 2.AG.GZ,$$

$$OB^2 = BG^2 + OG^2 - 2.BG.GZ;$$

$$\therefore P.OA^2 + Q.OB^2 = P.AG^2 + Q.BG^2 + (P+Q)O^2G,$$

for $P.AG = Q.BG$ by § 73, page 119;

$\therefore$ for two particles $N = \bar{N} + M.h^2$.

Next, let there be three particles, and suppose P, Q, and R to be their masses. Let A, B, and C be the projections of their respective positions. Let H be the projection of the C.G. of P, Q, and R. Then, by two applications of the preceding result, we have—

From P+Q at G and R at C,

$$(P+Q).OG^2 + R.OC^2 = (P+Q)GH^2 + R.CH^2 + (P+Q+R)OH^2.$$

$$\text{But } P.OA^2 + Q.OB^2 = P.AG^2 + Q.BG^2 + (P+Q).OG^2.$$

Add, cancel terms occurring on both sides, and remark that in ΔHAB

$$GH^2 + AG^2 - 2AG.GZ = HA^2, \text{ and } BG^2 + GH^2 + 2BG.GZ = HB^2;$$

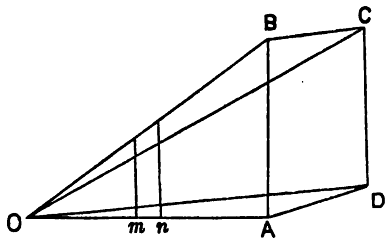
$$\therefore P.OA^2 + Q.OB^2 + R.OC^2 = P.HA^2 + Q.HB^2 + R.HC^2 + (P+Q+R).OH^2.$$

Similarly, by three applications of the preceding formula, we can show that the proposition is true where there are four particles; and so on universally.

133. Calculation of Moment of Inertia and Radius of Gyration of Simple Figures.

(1.) *If a rod of length* l *rotate about an axis through the end and perpendicular to the rod, then* $k^2 = \frac{1}{3} l^2$.

The calculation may be easily effected by means of the following auxiliary construction.



Let OA be the rod, and let ABCD be a square, the plane of which is perpendicular to the rod, and the side of which is equal to OA. Join O to the points A, B, C, D.

Let the rod be divided into very small equal parts, and through m and n, two adjacent points of division, pass planes parallel to ABCD.

The side of the square formed by any of these sections is equal to its distance from O.

The volume of the thin portion of the pyramid bounded by these planes is $Om^2.mn$.

Let s be the mass of a unit of length of the rod, then the mass of length mn is $mn.s$.

Now the moment of inertia of mn about the point O is $Om^2.mn.s$, that is, the volume of the corresponding portion of the pyramid multiplied by s.

Hence the whole moment of inertia of the rod is equal to the whole volume of the pyramid multiplied by s;

$$\text{or } N = \tfrac{1}{3}OM^2.OM.s$$
$$= \tfrac{1}{3}OM^2.\text{ mass of the rod};$$
$$\therefore k^2 = \tfrac{1}{3}l^2.$$

(2.) *For a rod about an axis through its centre* $k^2 = \frac{1}{12}l^2$.

From the formula

$$N = \overline{N} + M.h^2,$$

we have, if N refer to the end and $\overline{N}$ to the middle,

$$M.\frac{l^2}{3} = \overline{N} + M.\frac{l^2}{4};$$
$$\therefore \overline{N} = M.\frac{l^2}{12};$$
$$\therefore k^2 = \tfrac{1}{12}.l^2.$$

Another way.—Consider the rod made up of two each of length $\frac{1}{2}l$ and mass $\frac{1}{2}M$, and take the sum of the moments of inertia about the common extremity,

$$\therefore N = 2(\tfrac{1}{2}M)\tfrac{1}{3}(\tfrac{1}{2}l)^2 = M\frac{l^2}{12}.$$

(3.) *For a rectangle about an axis parallel to one of its sides.*

Let l_1, l_2, be the breadths of the two parts into which the axis divides the rectangle, and let M_1, M_2, be their masses.

Suppose the rectangle divided into thin rods perpendicular to the axis, then, by finding the moment of inertia of each and taking the sum, we have $N = \frac{M_1 l^2_1}{3} + \frac{M_2 l^2_2}{3}$.

But if M be the whole mass,

$$M_1 : M :: l_1 : l_1 + l_2,$$
$$M_2 : M :: l_2 : l_1 + l^2,$$

substituting the values of M_1 and M_2 from these equations, we have $N = \frac{M}{3}\,\frac{l^3_1 + l^3_2}{l_1 + l_2}$.

If $l_1 = l_2 = \frac{1}{2}l$, so that the axis passes through the centre;

$$\therefore N = M\frac{l^2}{12}.$$

(4.) *For a rectangle about an axis parallel to a side, and distant* h *from the centre:*

$$N = N + M.h^2,$$
$$= M\left(\frac{l^2}{12} + h^2\right).$$

(5.) *For a rectangular lamina,* a *by* b, *about an axis perpendicular to its plane through the centre.*

Find the moments of inertia N_1 and N_2 about axes through the centre perpendicular respectively to the sides a and b, then $N_1 = M.\frac{a^2}{12}$; $N_2 = M.\frac{b^2}{12}$,

$$\text{and } N = N_1 + N_2$$
$$= M\frac{a^2+b^2}{12}.$$

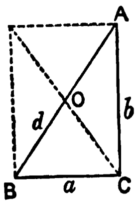
(6.) *For a parallelopiped, the sides of which are* a, b, *and* c, *about an axis through the centre parallel to* c.

By dividing the parallelopiped into rectangular lamina perpendicular to the axis, and summing their moments of inertia, we have moment of inertia $= \frac{1}{12}M(a^2+b^2)$.

(7.) *For a right-angled triangle about an axis through the Centre of Gravity and perpendicular to its plane.*

Complete the rectangle and draw the diagonals intersecting in O.

Let N be the moment of inertia of each of the triangles ABC, ABD about axes through the C.G. of the respective triangles, and let M be the mass of each. Let d be the length of each diagonal. Then the moment of inertia of each triangle about O is



$$N + M\left(\frac{d}{6}\right)^2.$$

But the moment of inertia of the whole rectangle about O is

$$\frac{1}{12}(2M)(a^2+b^2) \text{ or } \frac{1}{6}Md^2.$$

By equating these results,

$$\therefore\ 2\text{N}+2\text{M}\frac{d^2}{36}=\tfrac{1}{6}\text{M}d^2;$$

$$\therefore\ \text{N}=\frac{1}{18}\text{M}d^2=\frac{1}{18}\text{M}\,(a^2+b^2).$$

(8.) *For the same triangle about an axis perpendicular to its plane through the point A.*

The distance of the C.G. from A is $\tfrac{2}{3}\sqrt{\left(b^2+\frac{a^2}{4}\right)}$;

$$\therefore\ \text{from } \text{N}=\overline{\text{N}}+\text{M}h^2,$$

$$\text{we have } \text{N}=\tfrac{1}{18}\text{M}\,(a^2+b^2)+\text{M}\frac{4}{9}\left(b^2+\frac{a_2}{4}\right),$$

$$=\text{M}\left(\frac{a^2}{6}+\frac{b^2}{2}\right).$$

For an isosceles triangle of height b and base a about an axis through the vertex.

Dividing the triangle into two right-angled triangles like the last, we have

$$\text{N}=2\frac{\text{M}}{2}\left(\frac{a^2}{24}+\frac{b^2}{2}\right)$$

$$=\text{M}\left(\frac{a^2}{24}+\frac{b^2}{2}\right).$$

(9.) *For a regular polygon of* n *sides about an axis through the centre perpendicular to its plane.*

Divide the polygon into n isosceles triangles, the side of each being a, the side of the polygon and the height of each the radius r of the inscribed circle. The mass of each will be $\frac{\text{M}}{n}$.

The moment of inertia for each triangle will be

$$\frac{\text{M}}{n}\left(\frac{a^2}{24}+\frac{r^2}{2}\right);$$

$$\therefore\ \text{N}=\text{M}\left(\frac{a^2}{24}+\frac{r^2}{2}\right).$$

(10.) *For a circle about an axis through the centre perpendicular to its plane.*

We may suppose the circle obtained from the regular polygon by diminishing a without limit, keeping r constant.

Hence $N = \frac{1}{2}Mr^2$.

(11.) *For a circle about a diameter.*

Take two diameters at right angles, and apply the formula

$$N = N_1 + N_2.$$

Here N refers to an axis perpendicular to the plane, and is therefore $\frac{1}{2}Mr^2$, and N_1 and N_2 are equal by symmetry.

$$\therefore \tfrac{1}{2}Mr^2 = 2N_1, \text{ or } N_1 = \tfrac{1}{4}Mr^2.$$

(12.) *For a Fly-Wheel.*

If the inner and outer radii are respectively r_1 and r_2, and we suppose the wheel formed by taking a circular plate of mass M_1 from a larger concentric plate of mass M_2, then

$$N = \tfrac{1}{2}M_2 r_2^2 - \tfrac{1}{2}M_1 r_1^2,$$

$$\text{but } M_1 : M_2 :: r_1^2 : r_2^2;$$

$$\therefore M_2 - M_1 : M_2 :: r_2^2 - r_1^2 : r_2^2;$$

but $M_2 - M_1 = M$, the mass of the ring.

$$\therefore M_1 = \frac{Mr_1^2}{r_2^2 - r_1^2},$$

$$M_2 = \frac{Mr_2^2}{r_2^2 - r_1^2};$$

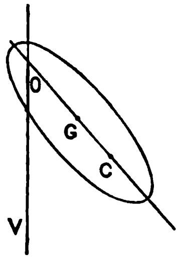
$$\therefore N = \tfrac{1}{2}M(r_2^2 + r_1^2).$$

Example 59.—The Compound Pendulum.—A body oscillates about a fixed horizontal axis under the action of gravity, to investigate the motion.

Let a vertical section through the centre of gravity G meet the axis in O.

Let the body start from rest when OG makes an angle a with the vertical OV, and let the angular velocity be ω when the angle between OG and OV is θ.

Let K be the radius of gyration of the body about o, and let h be the distance OG. Since, while the angle GOV changes from a to θ, the C.G. falls through a vertical height $h(\cos\theta - \cos a)$, the energy exerted $= \mathrm{W}h(\cos\theta - \cos a)$. Now the energy stored is $\frac{1}{2}\frac{\mathrm{W}}{g}\mathrm{K}^2\omega^2$;



$$\therefore \mathrm{W}h(\cos\theta - \cos a) = \frac{1}{2}\frac{\mathrm{W}}{g}\mathrm{K}^2\omega^2,$$

$$\text{Or } h(\cos\theta - \cos a) = \frac{\mathrm{K}^2\omega^2}{2g}.$$

By comparing this equation with that obtained on page 200 for a Simple Pendulum, it will be seen that the equations are identical if l, the length of the simple pendulum, is equal to $(\mathrm{K}^2 \div h)$.

Hence $\frac{\mathrm{K}^2}{h}$, or its equivalent $\frac{k^2+h^2}{h}$, is termed the length of the *isochronous simple pendulum.* Let C in the line OG be taken so that OC is the length of the isochronous simple pendulum, then C is termed the *centre of oscillation.*

The centres of oscillation and suspension are reciprocal; that is, if when the body is suspended from O the centre of oscillation is C, then when it is suspended from C the centre of oscillation will be O, the axes in the two cases being parallel.

In the first case, since $l = \frac{k^2+h^2}{h}$,

$$\therefore h(l-h) = k^2,$$

$$\text{or OG.CG} = k^2.$$

Now, if we make C the point of suspension, and let O′ be then the centre of oscillation, we obtain in a similar manner

$$\text{CG.O}'\text{G} = k^2.$$

Therefore O′ coincides with O.

The time of a complete small oscillation is (see page 202)—

$$t=2\pi\sqrt{\frac{l}{g}}=2\pi\sqrt{\frac{k^2+h^2}{hg}}.$$

Captain Kater's Pendulum.—This pendulum was designed to determine the value of g at different places. It consisted of a bar of metal, with two knife edges so placed that when one is the centre of suspension the other is nearly the centre of oscillation. A sliding weight which can be clamped at any point serves the purpose of adjusting the instrument until the times of oscillation when the pendulum is suspended on the two knife edges are exactly equal. The distance between the edges is then the length l. The time of an oscillation may be calculated with great exactness, and then all the quantities in the equation

$$t^2=\frac{4\pi^2 l}{g}$$

are known excepting g, which may therefore be calculated from the equation.

Example 60.—*A body whose weight is* P *draws up another body whose weight is* W *by means of a wheel and axle. Find the angular velocity of the wheel and axle when* P *has descended through a given space* s.

When P descends through a space s, let W ascend through a space r. Let a and b be the radii of wheel and axle respectively,

$$s:r::a:b.$$

Now the energy exerted $=$ Ps,

the work done $=$ Wr,

and the difference is stored in the moving parts of the machine.

Let v be the velocity of P after descending s.

" V be the velocity of W after ascending r.

" Q be the weight of the wheel and axle.

" k be the radius of gyration of wheel and axle.

" ω be the angular velocity required.

Then the energy stored

$$=\tfrac{1}{2}\frac{\mathrm{P}}{g}v^2+\tfrac{1}{2}\frac{\mathrm{W}}{g}\mathrm{V}^2+\tfrac{1}{2}\frac{\mathrm{Q}}{g}k^2\omega^2.$$

We have, however, a relation between v, V and ω, for evidently

$$\omega=\frac{V}{b}=\frac{v}{a};$$

$$\therefore\ Ps-Wr=\frac{1}{2g}(Pa^2+Wb^2+Qk^2)\omega,$$

$$s\cdot\frac{aP-bW}{a}=\frac{1}{2g}(Pa^2+Wb^2+Qk^2)\omega,$$

from which ω can be determined.

Combination of Motions of Translation and Rotation.—The total kinetic energy of a body which progresses and rotates is the sum of the energy due to the motion of translation and that due to the motion of rotation ; hence it is

$$\tfrac{1}{2}Mv^2+\tfrac{1}{2}MK^2\omega^2.$$

Example 61.—*A cylinder rolls down a rough inclined plane so that its axis remains always horizontal, find the velocity when the* C.G. *has descended through a vertical height* h.

Let v be the velocity of the C.G., and ω the angular velocity. By equating the energy exerted to the energy stored, we have

$$Wh=\frac{Wv^2}{2g}+\frac{Wk^2\omega^2}{2g}.$$

But if a be the radius,

$$v=a\omega,$$

$$\text{and } k^2=\tfrac{1}{2}a^2;$$

$$\therefore\ 2gh=v^2+\tfrac{1}{2}v^2,$$

$$v^2=\tfrac{2}{3}\text{ of }2gh.$$

EXERCISE XVII.

1. The moment of inertia of a cylinder about its axis is $\frac{1}{2}Mr^2$.

2. For a cylinder about a generating line $K^2=\frac{3}{2}r^2$.

3. Having given that for a sphere rotating about a diameter $k^2=\frac{2}{5}r^2$, prove that for a sphere rotating about an axis touching it $K^2=\frac{7}{5}r^2$.

4. For a cylinder of length h and radius r about a diameter of one end $K^2=\frac{1}{3}h^2+\frac{1}{4}r^2$. (To calculate the part $\frac{1}{3}h^2$, use an auxiliary figure as on page 235.)

5. In the Atwood's machine (see figure of page 54), if m be the mass of the pulley A, r its radius, and k its radius of gyration, show that neglecting the passive resistances and the weight of the cord the velocity v acquired by the mass P+Q when it has descended h feet is such that

$$v^2=\frac{2ghr^2Q}{(2P+Q)r^2+mk^2}.$$

6. If the pulley be a solid circular disc weighing $\frac{1}{2}$ pound, if P be 2 pounds, and Q $\frac{1}{2}$ pound, find the time in which the greater weight descends 18 feet, and find its acceleration.

7. In a wheel-and-axle, the radius of the wheel is 2 feet and its weight 1 pound, the radius of the axle 6 inches and its weight 3 pounds, the power 7 pounds and the weight 14 pounds; how long will the weight of 14 pounds be in ascending 30 feet?

8. Motion is communicated to a cylinder weighing 100 pounds moveable about a horizontal axis, by a weight of 10 pounds attached to a cord which is coiled on the cylinder. How far will the weight descend in 10 seconds?

9. What must be the weight, that when attached to the same cylinder it may move through 16·1 feet in 3 seconds?

10. If the cylinder weigh 20 pounds, and the weight 30 pounds, with what acceleration will the weight descend?

11. A sphere rolls down an inclined plane; show that the square of the velocity after rolling a given distance is $\frac{5}{7}$ of what it would be if the plane were smooth, so that the sphere would slide $(k^2=\frac{2}{5}r^2)$.

12. If a sphere rolls down a plane inclined at an angle whose sine is ·14, find the acceleration.

ANSWERS.

Exercise I. (Page 14.)

(1.) 888 feet. (2.) $v=48$, $f=8{\cdot}6$. (3.) 810. (4.) § 12, ii. (5.) 6 nearly. (6.) 24,000. (7.) 40 miles an hour. (8.) 76. (9.) $8\frac{3}{4}$ miles. (10.) Time $=8$ seconds, $f=5\frac{1}{3}$. (11.) $\dot{\cdot 4}$. (12.) 240 feet per second. (13.) 1740 feet. (14.) $t=\sqrt{51}$, or 7·2 seconds, distance from A $12\sqrt{51}$ feet. (15.) 44. (16.) $\frac{11}{120}$ seconds. (17.) 8. (18.) $(vlt' \div l't.)$ (19.) $(flt'^2 \div l't^2.)$ (20.) $\frac{1}{72}$. (21.) 1·2. (22.) $\frac{3}{88}$ seconds. (23.) $67\frac{1}{2}$ miles.

Exercise II. (Page 28.)

(1.) 130·23 feet per second; distance 273·32 feet. (2.) 481 feet per second. (3.) 145·94 feet per second. (4.) $65\frac{2}{3}$ or $22\frac{1}{3}$. (5.) $\frac{5}{6}$ mile 10 minutes. (6.) N.N.E. $60\sqrt{(2+\sqrt{2})}$. (7.) 3250 metres. (8.) 18 seconds. (9.) $\frac{63}{88}$ degrees per second. (11.) Height $=36$ feet, time $=3$ seconds, range $=75$ feet. (12.) The square of the velocity $=$

$$\frac{m^2v^2+m'^2v'^2+2mm'vv'\cos a}{(m+m')^2.}.$$

Exercise III. (Page 36.)

(1.) Height 225 feet, time $2\frac{1}{2}$ seconds or 5 seconds. (2.) 2 seconds nearly. (3.) $1\frac{80}{123}$ feet per second. (4.) Time $=\frac{1}{2}(1+\sqrt{17})$ or 2·56 seconds, velocity $=65{\cdot}9$. (5.) 402·5 feet, $v=177{\cdot}1$. (6.) $2g$ or 64 feet per second. (7.) 70·4. (9.) 299 feet. (10.) 320 feet. (11.) The lead is detached at height of 567 feet, and rises $102\frac{33}{64}$ feet

higher, or $669\frac{33}{64}$ feet in all. (12.) 1, 3, 5 feet. (13.) 3864. (14.) If $g=32$, the interval is $\frac{1}{4}(\sqrt{h}-\sqrt{h'})$. (15.) $3\frac{5}{8}$. (16.) $1\dot{3}$ seconds, $28\cdot\dot{4}$ feet. (17.) A requires more than 2 seconds to reach its highest point; hence, when A begins to descend, B is already falling lower down with an acquired velocity which increases at the same rate as that of A; hence, A can never overtake B. (19.) $3g$. (20.) $4\cdot155g$. (21.) 48. (22.) $8\frac{1}{21}$ feet.

Exercise IV. (Page 44.)

(1.) $V\sin\alpha - gt$. (2.) $v^2=3856-1920\sin30°$; or, $v=53\cdot8$. (3.) 91 feet, $14\frac{1}{16}$ feet. (4.) $\dfrac{V^2\sin^2 45}{2g}=125$; $\therefore V=40\sqrt{10}=125\cdot7$; $R=500$; $T=5\cdot6$ seconds. (5.) 1156. (6.) $25\frac{5}{32}$. (7.) $\tan\alpha=3\cdot6$, $V=242$, $ht=900$. (10.) $V\sin\alpha$. (11.) 158 feet. (13.) $\tan\alpha=\frac{4}{3}$. (14.) $V=81$, $H=249\cdot5$ feet, $\tan\theta=\cdot685$. (15.) $V=645$ feet $\tan\alpha=\frac{25}{176}$. (16.) $4608\sin(\alpha-30°)\cos\alpha\div3g$. (17.) $60°, \dfrac{384}{g}$. (18.) $6\sqrt{2g}$.

Exercise V. (Page 68.)

(1.) $2\frac{2}{45}$. (2.) $3\cdot2$. (3.) 1250 pounds and 625 pounds. (4.) $25\cdot6$ feet. (5.) 128 ounces. (6.) 50 ounces 16 feet per second. (7.) 32 feet. (8.) $7\frac{1}{2}$ pounds. (9.) 8 feet. (10.) 100 feet. (11.) 12 seconds. (12.) $3\frac{1}{13}$ feet per second. (13.) $15\frac{5}{13}$ feet. (14.) 5. (16.) $48\cdot3$. (17.) $85\frac{5}{7}$ tons. (18.) $55\frac{5}{8}$ and $63\frac{1}{8}$ ounces. (19.) $\frac{4}{7}g$. (20.) $\frac{1}{2}\sqrt{35}$ or $2\cdot958$ seconds. (21.) 600, unit of mass being 1 cwt. (22.) 15 seconds. (23.) 4 inches. (24.) $\dfrac{am'^2f}{a'm^2}$. (25.) $\sqrt{6}$ or $2\cdot449$ seconds. (26.) The unit of surface is 484 yards, therefore the unit of length is 22 yards, or 66 feet. If 1 foot and 1 second be units, the acceleration of gravity is 32; if with the new unit this acceleration is $51\frac{2}{3}$, the new unit of acceleration is $32\div58\frac{2}{3}$ times the old; hence,

$$32\div58\tfrac{2}{3}=66\div t^2;$$
$$\therefore t=11 \text{ seconds.}$$

(27.) 8 feet per second is 2 feet per $\frac{1}{4}$ second, and 32 feet per second

in a second is 2 feet per $\frac{1}{4}$ second in $\frac{1}{4}$ second; hence, unit of length is 2 feet.

$$m' = v'd' = l'^3 d';$$
$$\therefore 2240 = 8d',$$
$$d' = 280,$$

but d' is the density compared with a substance, a cubic foot of which weighs 1 pound; therefore density compared with water is $280 \div 62\frac{1}{2}$, or 4·48. (28.) For the two system of units we have

Length.	Time.	Mass.	Force.
1 foot.	1 second.	1,000 ounces.	1,000 ounces $\div g$.
x feet.	x seconds.	13,500x^3 ounces.	12,000 ounces.

$$\therefore l' = x,\ t' = x,\ m' = 13{\cdot}5\ x^3,\ F' = 12g,$$
$$\text{but } F' = a'm' = \frac{l'm'}{t'^2},$$
$$\therefore 12 \times 32 = \frac{x \times 13{\cdot}5x^3}{x^2},$$
$$x = 5\tfrac{1}{3}.$$

Exercise VI. (Page 87.)

(3.) 22·5. (5.) $12\frac{1}{2}$. (6.) Two balls of the same size meet equal resistance from the air, but if they are projected with the same velocity, the heavier will have the greater energy, and will therefore overcome that resistance the further. (7.) $6\frac{29}{56}$ feet per second. (8.) 25,344,000 foot-pounds. (9.) 216 feet. (10.) Distance $\frac{1}{2}g$ or 16·1 feet, time 2 seconds. (11.) 2·3 tons. (12.) 36·2 feet. (13.) $5\frac{164}{167}$. (14.) 468·75 feet. (15.) $1\frac{1}{2}$ pounds. (16.) $1885\frac{5}{7}$ seconds. (17.) In both cases 215·1 feet, $7\frac{1}{3}$ seconds. (18.) 137·5 seconds, 14·6 feet per second. (19.) Time of fall $= \frac{3}{4}$ seconds, and velocity 24; hence the momentum, before and after the shock, $= 48 - 24$. Therefore common velocity immediately after $= 8$, and as the acceleration is $\frac{1}{3}g$ for the time of ascent, we have

$$9 = 8t + \tfrac{1}{3}.16t^2,$$
$$\therefore t = \tfrac{3}{4},$$
$$\therefore \text{the whole time} = 1\tfrac{1}{2} \text{ seconds.}$$

(20.) 1 in 50. (21.) 128. (22.) 131·648. (23.) $13\frac{11}{28}$ miles an hour. (24.) $v^2 = 2\frac{2}{3}g$.

Exercise VII. (Page 96.)

(1.) (*a.*) 100; (*b.*) 120; (*c.*) 100; (*d.*) 116·6; (*e.*) 75. (2.) The weight having the horizontal motion of the carriage, the direction and tension of the string in the first case will be the same as if the suspension took place in a carriage at rest. In the second case the string remains vertical, but the tension is zero. (3.) The velocity of the body perpendicular to the direction of the well is unaltered, but the parts of the well itself move with less and less velocity as we descend, hence the body will strike the well on the side towards the direction of the earth's motion, that is to say, on the east. (4.) $7\frac{1}{2}$ seconds. (5.) 90 feet. (6.) 150 pounds. (7.) 1207·5 yards, taking $g = 32{\cdot}2$.

Exercise VIII. (Page 109.)

(1.) 3 and 4 tons. (3.) $20\sqrt{2}$ on B, and $20\sqrt{(2-\sqrt{2})}$ on A and C. (4.) 37·5899. (5.) 38 pounds. (7.) 2 pounds parallel to AC through the middle of BC. (8.) $2\sqrt{2}$ bisecting the angle between 3 and 4. (9.) 19·5. (11.) 97. (12.) $5\sqrt{2}$ pounds. (13.) 3·52 pounds, 9·36 pounds. (15.) Result of five forces $=\sqrt{5}$ lbs. (16.) 148·6 inches. (17.) and (19.) See Example 33. (20.) If θ be inclination of the beam, $\cos\theta = \frac{2}{3}$, and pressure $= 75$ pounds. (21.) One-sixth of the circumference apart. (22.) See page 82 (iv.) $\sin\alpha = 6$, $\cos\beta = \frac{15}{17}$. (24.) Resolving in the direction of the plane $2\sin\theta = \cos\theta + \theta 1 : 4\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin^2\frac{\theta}{2}$; or $\tan\frac{\theta}{2} = 2$. (25.) Thrust in AD $= 13$ cwt., thrust in CD $= 5$ cwt., tension in AB $= 5$ cwt.

Exercise IX. (Page 124.)

(1.) 1·44 feet. (2.) The cord and rod are each inclined at 30° to the horizon, and at 60° to one another, and the pressure at 60° to the horizon. (3.) $6\frac{3}{4}$ and $1\frac{1}{4}$ feet. (4.) 1 foot and $\frac{3}{4}$ foot. (5.) 2 inches from one end. (6.) 5 feet. (7.) AC $= 85$ inches, BC $= 152{\cdot}26$ inches; tension in AC, 6·46 pounds; in BC, 4·96 pounds. (8.) 148·6 inches. (9.) 60 inches and 25 inches. (11.) 17·8. (12.)

The sines of the angles a, b, c between the strings are determined from the equations—

$$\frac{\sin a}{4}=\frac{\sin b}{5}=\frac{\sin c}{6},$$

$$\text{and } a+b+c=360^\circ.$$

From which we obtain—

$$\cos a=-\cdot 75, \cos b=-\cdot 5626, \cos c=-\cdot 125.$$

(13.) 2 inches from the point. (14.) 11·825, if a=the weight per foot, the strain$=3a$. (15.) 1. Reaction of wall, 90 pounds; 2. Tension of string, 90 pounds; 3. Reaction of plane, 112 pounds. 4. Weight of beam, 112 pounds. (17.) $13\frac{1}{3}\sqrt{3}$, or 23·094 feet. (18.) $615^2:728^2$. (19.) If θ be the inclination of the beam, β the angle between the beam and string, $\cos\theta=\frac{2}{3}\sin\beta$. (20.) AO=10·5, BO=8·4, the vertical through O bisects the angle O, if this angle is $2a$, then the tension=45 pounds$\div\cos a$=100·14 pounds. (21.) Taking moments about A, $\therefore \frac{1}{2}W=R$.

Resolving horizontally, $\therefore R.\cos 30=T$.

$$\therefore T=\frac{\sqrt{3}}{4}W.$$

(22.) The rod will make an angle with the vertical $=2\angle B$. In the second case the point of suspension will be 44 inches from A.

Exercise X. (Page 134.)

(5.) Take moments about the centre of the circle. (7.) $Wb=Pa$. The left-hand side being constant, if a diminishes P increases, and therefore P + W, or the pressure, increases. Regarding the man and things carried as one body, we see that the change only affects the mutual actions at the hand and shoulder, and not the pressure on the ground. (8.) If C be the scale and AC the rod, there is a pressure P at A along CA, and an equal pressure at C along AC. Resolve into horizontal and vertical components HV. VV is a couple having a moment V.SF – V.AF or V.AS, tending to turn the beam towards C. The component H at A is counteracted by the reaction of F, but H at C tends to push out the scale. Thus both components cause C to descend. The horizontal string

would counteract the lower H, but would not affect the action of the couple VV.

EXERCISE XI. (Page 149.)

(1.) $R_1 = R_3 = 31{\cdot}4$, $R_2 = 33{\cdot}8$, $R_4 = 46$ ounces. (2.) Let R be the pressure between the upper sphere and the shell, R_2 that between the spheres, R_3 that between the lower sphere and the base of the shell, R_4 that between the lower sphere and the curved part of the shell; then

$$R = \frac{W}{\sqrt{3}},\ R_2 = \frac{2W}{\sqrt{3}},\ R_3 = \frac{3W}{\sqrt{3}},\ R_4 = \frac{4W}{\sqrt{3}}.$$

(4.) 160 pounds. (5.) 8 feet. (6.) $15\frac{15}{16}$ ounces. (7.) 2 pounds and 10 pounds. (8.) 9·6 ounces on each. (9.) On the pan containing the 8 ounces the pressure is $9\frac{1}{3}$ ounces, and on the other 10 ounces. (10.) $15\frac{15}{31}$, $30\frac{30}{31}$ ounces. (11.) 20 and 60. (12.) $23\frac{9}{17}$ feet. (13.) $4\frac{11}{41}$ pounds and $29\frac{36}{41}$ pounds. (14.) $F = Qg\cos\alpha(Q\sin\alpha - P) \div (P + Q)$. (15.) Let the angle between the rods be 2α, and the angle between the string and rod be θ. Resolving vertically the forces acting on the weight 2W;

$$\therefore T.\cos(\theta + \alpha) + W = 0.$$

Resolving along the rod the forces acting on a ring,

$$\therefore W\cos\alpha = T\cos\theta;$$

$$\therefore \tan\theta = \frac{1 + \cos^2\alpha}{\sin\alpha\cos\alpha};$$

$$\therefore T = \frac{W\sqrt{(1 + 3\cos^2\alpha)}}{\sin\alpha}.$$

(16.) $T = 2w$. (18.) Each $= 2w$. (19.) $\frac{1+54g}{54g}$, $\frac{1+54g}{27g}$. (20.) $17\frac{1}{2}$ pounds. (22.) 32·1. (23.) Rod $86\frac{2}{9}$ inches, $\tan\alpha = \frac{4}{9}$. (24.) 24 pounds. (25.) If W be the weight, and $2l$ the length of the rod, and θ the inclination, and if w be the weight of the ring, and a the length of the string

$$\cos\theta = \frac{-Wl \pm \sqrt{(W^2l^2 + 32a^2w^2)}}{8aw}.$$

(26.) Call the length of each rod 2, the angle between the cord

and rod ϕ, the reaction at B or C, R, and then take moments about A, and resolve vertically ;

$$\therefore\ 2\text{R} \cos\theta - w \cos\theta = 4\text{T} \sin\phi \text{ and } \text{R} = w.$$

If l be the length of each string, $l^2 = 1 + 8 \cos^2\theta$.

$$\text{Also, } \frac{\sin\phi}{\sin 2\theta} = \frac{1}{l}\ ;$$

$$\therefore\ \frac{\text{T}}{w} = \frac{\cos\theta}{4 \sin\phi} = \frac{l \cos\theta}{4 \sin 2\theta} = \tfrac{1}{8} \sqrt{(\text{cosec}^2\theta + 8 \cot^2\theta)}.$$

Exercise XII. (Page 156.)

(1.) $\frac{1}{3}v$. (2.) 6·75 feet, 5·5 and 7·5 feet, 6·33 feet and 7 feet. (3.) $6\frac{17}{19}$, $1\frac{2}{19}$, $1\frac{17}{19}$. (4.) 0 and 20 feet, 12 and 32 feet. (5.) -3 feet, -13 and $+3$ feet, $-6·33$ and -1 foot. (6.) 37, 11. (7.) $38\frac{1}{4}$ feet. (8.) $\frac{1}{3}$. (9.) $e = \frac{1}{6}$, B$=$6A. (10.) 30°.

Exercise XIII. (Page 172.)

(1.) 12 inches from the larger weight. (2.) 11 inches. (3.) 6 inches. (4.) $\frac{1}{3}$ each. (5.) $\frac{2}{9}$ of the side. (6.) $\frac{1}{9}$ of the side from the centre. (7.) $7\frac{3}{7}$ inches. (8.) 6 inches from the side $4-5$, and $4\frac{2}{3}$ inches from the side $5-6$. (9.) 3300 foot-pounds. (10.) 1560 foot-pounds. (11.) In the line of centres, $\frac{1}{6}$ of the radius of the larger circle from its centre. (12.) In the line joining the middle points of the arms at $\frac{2}{3}$ of its length from the shorter. (13.) 52 inches. (15.) W$(\sqrt{2}+1)$. (16.) 20,160 foot-pounds. (17.) $2\frac{1}{7}$ feet from end. (18.) $\text{Tan}\,a = 2\frac{1}{16}$. (19.) $4\frac{1}{2}$ inches from the base. (20.) $5\frac{1}{7}$ inches from the base. (26.) C.G. of a hemisphere is $\frac{3}{8}$ of radius from the centre $\sqrt{3}.r$.

Exercise XIV. (Page 202.)

(1.) g being 32, the pressure equals the weight of $401\frac{73}{135}$ pounds. (2.) $2\frac{1}{28}$ inches. (3.) $\frac{100\pi^2}{g}$. (4). $\frac{8\pi^2}{5g}$. (5.) $\frac{\pi\sqrt{2}}{\sqrt{g}}$. (6.) The C.G.

of P and Q. (7.) 16·1 feet. (8.) Equal to the weight of $2(3-\sqrt{3})$ or 2·535 pounds. (9.) See example 5, page 196. (10.) 100 foot-pounds. (11.) $154\frac{2}{7}$ feet, $\frac{288}{385}$. (13.) To be found approximately by calculating the pressures at different heights and approximating to the area of the curve of work. This area is a little more than 820, and velocity therefore a little more than 54.

$$(14.)\quad 80\times\frac{641^2}{640^2}=80\left(1+\frac{1}{640}\right)^2=80\left(1+\frac{1}{320}\right)\text{ nearly}=80\tfrac{1}{4}.$$

(15.) $\dfrac{mv^2}{2a}$.

EXERCISE XV. (Page 221.)

(1.) $2\frac{1}{2}$ feet from the end bearing the weight, the pressure $=6$ pounds. (2.) 4 pounds. (4.) 147 grammes. (5.) 15 kilogrammes. (6.) 8. (7.) The larger weight $\dfrac{3g}{86}$ the smaller $\dfrac{2g}{43}$. (8.) 6 tons. (9.) $37\frac{1}{2}$ pounds. (10.) $\sqrt{\frac{20}{41}g}$ and $6\sqrt{\frac{20}{41}g}$. (11.) $110\sqrt{\dfrac{2}{g}}$ seconds, $\dfrac{2\sqrt{2g}}{11}$ feet per second. (13.) $\frac{5}{24}$ pounds. (14.) ·7 pounds. (15.) 1800 pounds. (17.) $v=\sqrt{.6g}$ and $2\sqrt{.6g}$, $t=8\sqrt{15}\div\sqrt{g}$. (18.) $v=\sqrt{\dfrac{2g}{19}}\ t=10\sqrt{\dfrac{38}{g}}$. (19.) $129\frac{23}{95}$ pounds. (20.) $\sqrt{\frac{4}{577}}$, $\frac{2}{577}$, and 480 times these. (21.) $t^2=\dfrac{12970^2}{7440}$ or $t=22663$ seconds.

EXERCISE XVI. (Page 229.)

(2.) Equal to the angle of repose. (3.) If $\theta=$ the inclination of the base to the horizon, $\sin\theta=\dfrac{8\mu(1+\mu)}{3(1+\mu^2)}$. (4.) $7\frac{1}{12}$ feet. (5.) $\frac{1}{8}$. (6.) $100\,\dfrac{\sin a\pm\cos a}{\cos\beta+\sin\beta}$. (7.) Resolve horizontally and vertically and take moments about the foot. *Ans.*—$\frac{19}{39}$ of the length. (8.) Let a be the inclination of the plane, W the weight of the body;

then the least force is W $\sin 2a$, and it acts at an inclination a to the plane. (9.) $\operatorname{Tan}\theta = \frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}$. (10.) $\frac{1}{37}$.

EXERCISE XVII. (Page 242.)

(6.) $t = 3\sqrt{\frac{38}{g}}\,f = \frac{2g}{19}$. (7.) $t = 60\sqrt{\frac{19}{105g}}$ seconds. (8.) $298\frac{1}{3}$ feet. (9.) $6\frac{1}{4}$ pounds. (10.) 24·15 feet per second per second. (12.) $\frac{1}{10}g$.

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